2.7: Set \( I = [0, 1] \), and \( \Omega = \{ f \in C(I) : \text{Lip}(f) \leq 1, \int f = 0 \} \).

We will use the Arzelà-Ascoli theorem, of course.

The Lipschitz condition implies that \( \Omega \) is equicontinuous. (To prove this, fix any \( \varepsilon > 0 \). Set \( \delta = \varepsilon \). Then for any \( f \in \Omega \), and \( |x - y| < \delta \), we have \( |f(x) - f(y)| \leq \text{Lip}(f) |x - y| \leq |x - y| < \varepsilon \).

To prove that \( \Omega \) is bounded, note that if \( \int f = 0 \), and \( f \) is continuous, then there must exist an \( x_0 \in I \) such that \( f(x_0) = 0 \). Then for any \( x \in I \) and any \( f \in \Omega \), we have \( |f(x)| = |f(x) - f(x_0)| \leq \text{Lip}(f) |x - x_0| \leq |x - x_0| \leq 1 \). So \( ||f||_u \leq 1 \).

Finally we need to prove that \( \Omega \) is closed. Let \( (f_n) \) be a Cauchy sequence in \( \Omega \). Since \( C(I) \) is complete, there exists an \( f \in C(I) \) such that \( f_n \to f \) uniformly. We need to prove that \( f \in \Omega \). Since \( f_n \to f \) uniformly, we know both that \( \text{Lip}(f) \leq \lim \sup_{n \to \infty} \text{Lip}(f_n) \leq 1 \), and that \( \int f = \lim_{n \to \infty} \int f_n = 0 \). This proves that \( f \in \Omega \).
2.8: We will explicitly construct a dense countable subset $\Omega$ of $C([a, b])$. Without loss of generality, we can assume that $a = 0$ and that $b = 1$.

For $n = 1, 2, \ldots$, and for $j = 0, 1, 2, \ldots, n$, set $x_j^{(n)} = j/n$. Let $\Omega_n$ denote the subset of $C(I)$ of functions that (1) are linear on each interval $[x_{j-1}^{(n)}, x_j^{(n)}]$, and (2) take on rational values for each $x_j^{(n)}$. Since each function in $\Omega_n$ is uniquely defined by its values on the $x_j^{(n)}$’s, we can identify $\Omega_n$ by $\mathbb{Q}^{n+1}$. Hence $\Omega_n$ is countable.

Set $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Since each $\Omega_n$ is countable, $\Omega$ is countable.

It remains to prove that $\Omega$ is dense in $C(I)$. Fix any $f \in C(I)$, and any $\varepsilon > 0$. Since $I$ is compact, $f$ is uniformly continuous on $I$ so there exists a $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/2$. Pick an $n$ such that $1/n < \delta$, and pick a $\varphi \in \Omega_n$ such that $|\varphi(x_{j-1}^{(n)}) - f(x_{j-1}^{(n)})| < \varepsilon/2$ for $j = 0, 1, 2, \ldots, n$. We will prove that $||\varphi - f||_u < \varepsilon$: Fix an $x \in I$. Then pick $j \in \{1, 2, \ldots, n\}$ so that $x \in [x_{j-1}^{(n)}, x_j^{(n)}]$. Since $\varphi$ is linear in this interval, there is a number $\alpha \in [0, 1]$ such that

$$\varphi(x) = \alpha \varphi(x_{j-1}^{(n)}) + (1 - \alpha) \varphi(x_j^{(n)}).$$

Now

$$|f(x) - \varphi(x)| = |\alpha f(x) + (1 - \alpha) f(x) - \alpha \varphi(x_{j-1}^{(n)}) - (1 - \alpha) \varphi(x_j^{(n)})|$$

$$\leq \alpha |f(x) - \varphi(x_{j-1}^{(n)})| + (1 - \alpha) |f(x) - \varphi(x_j^{(n)})|.$$ 

Since $|f(x) - f(x_{j-1}^{(n)})| \leq \varepsilon/2$ (by the uniform continuity) and since $|f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})| < \varepsilon/2$ (by the choice of $\varphi$), we have

$$|f(x) - \varphi(x_{j-1}^{(n)})| \leq |f(x) - f(x_{j-1}^{(n)})| + |f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})| < \varepsilon.$$ 

Analogously,

$$|f(x) - \varphi(x_j^{(n)})| \leq |f(x) - f(x_j^{(n)})| + |f(x_j^{(n)}) - \varphi(x_j^{(n)})| < \varepsilon.$$ 

Together, (1), (2), and (3) imply that $|f(x) - \varphi(x)| < \varepsilon$. 
2.9: (a) Suppose that \( w(x) > 0 \) for \( x \in (0, 1) \). Then \( \| \cdot \|_w \) is a norm since:

(i) \( \| \lambda f \|_w = \sup_x w(x)|\lambda f(x)| = |\lambda| \sup_x w(x)|f(x)| = |\lambda| \| f \|_w \).

(ii) \( \| f + g \|_w = \sup_x w(x)|f(x) + g(x)| \leq \sup_x w(x)(|f(x)| + |g(x)|) \leq \sup_x w(x)|f(x)| + \sup_x w(x)|g(x)| = \| f \|_w + \| g \|_w \).

(iii) If \( f = 0 \), then clearly \( \| f \|_w = 0 \). Conversely, if \( f \neq 0 \), then \( f(x_0) \neq 0 \) for some \( x_0 \in (0, 1) \). Then \( \| f \|_w \geq w(x_0)|f(x_0)| > 0 \).

(b) Assume that \( w(x) > 0 \) for \( x \in [0, 1] =: I \). Set \( m = \inf_{x \in I} w(x) \) and \( M = \sup_{x \in I} w(x) \). Since \( I \) is compact and \( w \) is continuous, \( w \) attains both its inf and its sup, and therefore \( m > 0 \) and \( M < \infty \). Then

\[
\| f \|_u = \sup_{x \in I} |f(x)| \geq \sup_{x \in I} \frac{w(x)}{M} |f(x)| = \frac{1}{M} \| f \|_w.
\]

and

\[
\| f \|_u = \sup_{x \in I} |f(x)| \leq \sup_{x \in I} \frac{w(x)}{m} |f(x)| = \frac{1}{m} \| f \|_w.
\]

It follows that

\[
\frac{1}{M} \| f \|_w \leq \| f \|_u \leq \frac{1}{m} \| f \|_w.
\]

(c) Set \( \| f \| = \sup_{x \in I} |xf(x)| \). We will prove that \( \| \cdot \| \) is not equivalent to the uniform norm. Set for \( n = 1, 2, \ldots \)

\[
f_n(x) = \begin{cases} 1 - nx & x \in [0, 1/n], \\ 0 & x \in (1/n, 1]. \end{cases}
\]

Then \( \| f_n \|_u = 1 \) for all \( n \), while \( \| f_n \| = \sup_x x|f_n(x)| \leq 1/n \). This proves that there cannot be a finite \( M \) such that \( \| f \|_u \leq M \| f \|_w \) for all \( f \).

(d) We will prove that the set \( C(I) \) equipped with the norm \( \| \cdot \| \) is not a Banach space by constructing a Cauchy sequence with no limit point in \( C(I) \). For \( n = 1, 2, \ldots \), define \( f_n \in C(I) \) by

\[
f_n(x) = \begin{cases} x^{-1/2} & x \in (1/n, 1], \\ \sqrt{n} & x \in [0, 1/n]. \end{cases}
\]

Fix a positive integer \( N \). Then, if \( m, n \geq N \), we have

\[
\| f_n - f_m \| = \sup_{x \in [0, 1/N]} x|f_n(x) - f_m(x)|
\]

\[\leq \sup_{x \in [0, 1/N]} (x|f_n(x)| + x|f_m(x)|)
\]

\[\leq \sup_{x \in (0, 1/N)} (x \cdot x^{-1/2} + x \cdot x^{-1/2}) = 2N^{-1/2}.
\]

Consequently, \( (f_n)_{n=1}^\infty \) is a Cauchy sequence.

Now suppose that \( (f_n) \) converges with respect to the \( \| \cdot \|_w \) norm to some function \( f \in C(I) \). Then for any \( x \in I \) we have,

\[
\| f - f_n \| \geq x|f(x) - f_n(x)|
\]

Take the limit as \( n \to \infty \) to get

\[
0 \geq x|f(x) - x^{-1/2}|.
\]

It follows that \( f(x) = x^{-1/2} \) whenever \( x \neq 0 \), and consequently \( \| f \|_u = \infty \). In other words, \( f \notin C(I) \). (We do not know anything about \( f(0) \), but that is OK.)
Problem 1:

(d) The set $\Omega$ can contain a single function, for instance $f(x) = \sin(1/x)$. Note that for any fixed $x$, you can find a $\kappa$ (say $\kappa = x/2$) such that $f'(x)$ is bounded on $[x - \kappa, x + \kappa]$. But this does not imply that $f$ is uniformly continuous.

(e) Fix $a \in (0,1)$ and set $f_n(x) = n(x - a)^2$. Then $f'_n(a) = 0$ for all $n$. But $\{f_n\}_{n=1}^{\infty}$ is not equicontinuous at $a$. (Prove this!)

(f) Note of the conditions imply that $\Omega$ is bounded. You could for instance have the sequence of constant functions $f_n(x) = n$. Then $f'_n(x) = 0$ for all $n$ and all $x$ so all conditions are satisfied. But $\Omega = \{f_n\}_{n=1}^{\infty}$ is not a bounded set (with respect to the uniform norm).