Problem 1: Set $I = (0, 1)$ and let $(f_n)_{n=1}^\infty$ be a sequence of continuously differentiable functions on $I$. Set $\Omega = \{f_n : 1 \leq n < \infty\}$.

(a) For a given $n$, suppose that 
$$ \sup_{x \in I} |f_n'(x)| < \infty. $$
Prove that then $f_n$ is uniformly continuous.

(b) Suppose that 
$$ \sup_{x \in I} \sup_{1 \leq n < \infty} |f_n'(x)| < \infty. $$
Prove that then $\Omega$ is uniformly equicontinuous.

(c) Suppose that for every $x \in I$, there exists a $\kappa > 0$ such that 
$$ \sup_{1 \leq n < \infty} \sup_{y \in B_\kappa(x)} |f_n'(y)| < \infty. $$
Prove that then $\Omega$ is equicontinuous.

(d) Give an example of a set $\Omega$ of functions satisfying the condition in (c) that is not uniformly equicontinuous.

(e) Suppose that for a given $x \in I$, it is the case that 
$$ \sup_{1 \leq n < \infty} |f_n'(x)| < \infty. $$
Prove that $\Omega$ is not necessarily equicontinuous at $x$.

(f) Which, if any, of the examples listed in (a) – (e) represent a bounded set $\Omega$?
Extra problem on completeness: Fix a real number \( r \in (1, \infty) \), and let \( X \) denote the set of all real-valued sequences \( x = (x_1, x_2, x_3, \ldots) \) such that
\[
\sum_{n=1}^{\infty} |x_n|^r < \infty.
\]

(a) Fix a real number \( p \in [1, r) \). Define for \( x \in X \), the function
\[
f(x) = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.
\]
Show that \( f(x) = \infty \) for some \( x \in X \), and therefore \( f \) cannot be a norm on \( X \).

(b) Fix a real number \( p \in (r, \infty) \). Define for \( x \in X \), the function
\[
f(x) = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}.
\]
Show that \( f(x) \) is finite whenever \( x \in X \), and that \( ||x|| = f(x) \) defines a norm on \( X \). (Recall that you have already proven that \( f(x+y) \leq f(x) + f(y) \) in an earlier homework.) Show that \( (X, || \cdot ||) \) is not a Banach space.

(c) Repeat exercise (b) for the function
\[
f(x) = \sup_{n=1, 2, 3, \ldots} |x_n|.
\]

Another problem on completeness: Set \( I = [-1, 1] \) and set
\[
B = \{ f \in C(I) \cap C^1(I) : \sup_{x \in I} |f(x)| \leq 1 \text{ and } \sup_{x \in I} |f'(x)| \leq 1 \}.
\]
Consider the function \( d : X \times X \to [0, \infty) \) given by
\[
d(f, g) = \sup_{x \in I} |f(x) - g(x)|.
\]
Show that \( (B, d) \) is a metric space that is not complete.