Homework set 4 — APPM5440 Fall 2016 — partial solutions

**Solution for 2.4:** Let’s consider \( X = [-1, 1] \) instead. Then set \( f(x) = |x| \), and

\[
f_n(x) = \frac{1 + nx^2}{\sqrt{n + n^2x^2}}.
\]

Then \( f_n \to f \) uniformly, \( f_n \in C^\infty(X) \), and \( f \) is not differentiable. (To justify the shift we made initially, simply note that if we define \( g_n \in C([0, 1]) \) by \( g_n(y) = f_n(2y - 1) \), then \( g_n \) is an answer to the original problem.)

**Solution for 2.5:** Set \( I = [a, b] \). Let \( (f_n)_{n=1}^\infty \) be a Cauchy sequence in \( C^1(I) \). Since

\[
||f_n - f_m||_u \leq ||f_n - f_m||_{C^1},
\]

the sequence \( (f_n) \) is Cauchy in \( C(I) \). Since \( C(I) \) is complete, there exists a function \( f \in C(I) \) such that \( f_n \to f \) uniformly.

Next set \( g_n = f'_n \). Then

\[
||g_n - g_m||_u = ||f'_n - f'_m||_u \leq ||f_n - f_m||_{C^1},
\]

so \( (g_n) \) is Cauchy in \( C(I) \). Therefore, there exists a function \( g \in C(I) \) such that \( g_n \to g \) uniformly.

It remains to prove that \( f \in C^1(I) \), and that \( f_n \to f \) in \( C^1(I) \). Fix any \( x \in I \), and any \( h \in \mathbb{R} \) such that \( x + h \in I \). Then

\[
\frac{1}{h} (f(x + h) - f(x)) = \lim_{n \to \infty} \frac{1}{h} (f_n(x + h) - f_n(x)) = \lim_{n \to \infty} \frac{1}{h} \int_0^h f'_n(x + t) \, dt = \lim_{n \to \infty} \frac{1}{h} \int_0^h g_n(x + t) \, dt.
\]

Now recall that uniform convergence on a finite interval implies convergence of integrals. Since \( g_n \to g \) uniformly, we find that

\[
\frac{1}{h} (f(x + h) - f(x)) = \frac{1}{h} \int_0^h g(x + t) \, dt.
\]

Since \( g \) is continuous, the limit as \( h \to 0 \) exists, and so

\[
f'(x) = \lim_{h \to 0} \frac{1}{h} (f(x + h) - f(x)) = \lim_{h \to 0} \frac{1}{h} \int_0^h g(x + t) \, dt = g(x).
\]

This proves that \( f \in C^1(I) \). To prove that \( f_n \to f \) in \( C^1(I) \), we note that

\[
||f - f_n||_{C^1} = ||f - f_n||_u + ||f' - f'_n||_u = ||f - f_n||_u + ||g - g_n||_u.
\]

By the construction of \( f \) and \( g \), it follows that \( ||f - f_n||_{C^1(I)} \to 0 \).