Problem 1.8: Let \((x_n)\) be a sequence in \(\mathbb{R}\), let \(C\) denote the set of cluster points of \((x_n)\). Set 
\[ M = \sup C, \quad y_k = \sup \{x_n : n \geq k\} \]
and recall that 
\[ \limsup_{n \to \infty} x_n = \lim_{k \to \infty} y_k. \]

Show that \(C\) is closed: We will prove that \(C\) is complete. (Since \(\mathbb{R}\) is complete, \(C\) is closed iff it is complete.) Let \((c_j)\) be a Cauchy sequence in \(C\), and set \(c = \lim c_n\). We need to prove that \(c \in C\).

Set \(n_0 = 0\). Then for \(n = 1, 2, 3, \ldots\), pick an \(n_j\) such that \(n_j - 1 < n_j\) and 
\[ |c - x_{n_j}| < \frac{1}{j} \]
(this is possible since each \(c_j\) is the limit of a subsequence of \((x_n)\)). Then 
\[ \limsup_{j \to \infty} |c - x_{n_j}| \leq \limsup_{j \to \infty} (|c - c_j| + |c_j - x_{n_j}|) \leq \limsup_{j \to \infty} (|c - c_j| + 1/j) = 0. \]
So \(x_{n_j} \to c\), and therefore \(c \in C\).

Show that \(\limsup_{n \to \infty} x_n \geq \max C\): Since \(M \in C\), we know there exists a sequence \((x_{n_j})\) such that \(x_{n_j} \to M\). Then for any \(k\), we have 
\[ y_k = \sup \{x_n : n \geq k\} \geq \sup \{x_{n_j} : n_j \geq k\} \geq M. \]
Now take the limsup as \(k \to \infty\) to get the desired inequality.

Show that \(\limsup_{n \to \infty} x_n \leq \max C\): Pick any \(\varepsilon > 0\). We know that for some \(k\), we have 
\[ y_k \leq M + \varepsilon \]
(since if this were not true, then \((x_n)\) would have at least one cluster point larger than \(M\)). Take the limsup as \(k \to \infty\) to show that \(\limsup_{n \to \infty} x_n \leq M + \varepsilon\). Since \(\varepsilon\) is arbitrary, we are done.

Problem 1.10: Prove that
\[ (1) \quad \limsup_{n \to \infty} \inf_{\alpha} x_{n,\alpha} \leq \inf_{\alpha} \limsup_{n \to \infty} x_{n,\alpha}, \]
and that
\[ (2) \quad \sup_{\alpha} \liminf_{n \to \infty} x_{n,\alpha} \leq \liminf_{n \to \infty} \sup_{\alpha} x_{n,\alpha}. \]

Solution: Set \(y_n = \inf_{\alpha} x_{n,\alpha}\). Then clearly 
\[ y_n \leq x_{n,\alpha}, \quad \forall \alpha. \]
Take the limsup of both sides:
\[ \limsup y_n \leq \limsup x_{n,\alpha}, \quad \forall \alpha. \]
Finally take the infimum over \(\alpha\), nothing that \(\limsup y_n\) does not depend on \(\alpha\):
\[ \limsup y_n \leq \inf_{\alpha} \limsup x_{n,\alpha}. \]
This relation is (1).

To prove (2), analogously set \(z_n = \sup_{\alpha} x_{n,\alpha}\). Then \(x_{n,\alpha} \leq z_n\) for all \(\alpha\). Take the liminf to get \(\liminf x_{n,\alpha} \leq \liminf z_n\), and finally take the sup over \(\alpha\) to get (2).
**Problem 2:** Suppose that \((x_n)_{n=1}^{\infty}\) and \((y_n)_{n=1}^{\infty}\) are Cauchy sequences in a metric space \((X, d)\). Prove that the sequence \((d(x_n, y_n))_{n=1}^{\infty}\) converges.

**Solution:** Set \(\alpha_m = d(x_m, y_m)\). Since \(\mathbb{R}\) is complete, all we need to prove is that \((\alpha_m)\) is a Cauchy sequence.

Fix any two natural integers \(m\) and \(n\). Via two applications of the triangle inequality, we obtain
\[
d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m).
\]
It follows that
\[
d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_n, y_m). \quad (3)
\]
An analogous argument shows that
\[
d(x_n, y_n) - d(x_m, y_m) \leq d(x_m, x_n) + d(y_n, y_m). \quad (4)
\]
Together, (3) and (4) imply that
\[
|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_n, y_m). \quad (5)
\]
Fix \(\varepsilon > 0\). Since \((x_n)\) and \((y_n)\) are Cauchy, there exist \(N_1\) and \(N_2\) such that
\[
m, n \geq N_1 \implies d(x_m, x_n) < \varepsilon/2, \quad (6)
\]
\[
m, n \geq N_2 \implies d(y_m, y_n) < \varepsilon/2. \quad (7)
\]
Set \(N = \max(N_1, N_2)\). Then (5), (6), (7) imply that
\[
m, n \geq N \implies |\alpha_m - \alpha_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]