Problem 1.3: Note that the desired inequality is equivalent to the following pair of inequalities:
\[
\begin{align*}
\{ &d(x, z) - d(y, z) \leq d(x, y) \\
&d(y, z) - d(x, z) \leq d(x, y) \}
\end{align*}
\]
Now prove each of the two inequalities in the pair above separately.

Problem 1.5: We will prove that if \((X, \| \cdot \|)\) is a NLS, then the function
\[
d(x, y) = \frac{\|x - y\|}{1 + \|x - y\|}
\]
defines a metric on \(X\). It is easy to verify that \(d\) is symmetric and is zero iff \(x = y\). The challenge is the triangle inequality. Observe that
\[
d(x, y) = f(\|x - y\|), \quad \text{where} \quad f(t) = \frac{t}{1 + t}.
\]
Since \(f\) is monotonically increasing, and since \(\|x - y\| \leq \|x - z\| + \|y - z\|\), we immediately find that
\[
d(x, y) = f(\|x - y\|) \leq f(\|x - z\| + \|y - z\|).
\]
Next, use the following lemma:

**Lemma:** Suppose that \(f : [0, \infty) \to [0, \infty)\) is a differentiable function that satisfies \(f(0) = 0\), \(f' \geq 0\), and \(f'\) is monotonically decreasing. Then \(f(a + b) \leq f(a) + f(b)\) for every non-negative \(a\) and \(b\).

**Proof:** We have
\[
f(a + b) = f(a) + \int_a^{a+b} f'(t) \, dt \leq f(a) + \int_0^b f'(t) \, dt = f(a) + f(b) - f(0) = f(a) + f(b),
\]
where the inequality holds true since \(f'\) is positive but decreasing (so for every \(t\) we have \(f'(t) \leq f'(a + t)\).)

Since our \(f\) satisfies this property, we immediately get
\[
d(x, y) = f(\|x - y\|) \leq f(\|x - z\| + \|y - z\|) \leq f(\|x - z\|) + f(\|y - z\|) = d(x, z) + d(z, y).
\]

Problem 2: (a) The putative norms \(a, d, e, \) and \(f\) are norms. (\(b\) and \(g\) are semi-norms, \(c\) does not satisfy \(\|\alpha f\| = |\alpha| \|f\|\).)

(c) Set \(I = [0, 1]\) and consider the set \(X\) consisting of all continuous functions on \(I\), with the norm
\[
\|f\| = \int_0^1 |f(x)| \, dx.
\]
Prove that the space \(X\) is not complete.

**Solution:** A straight-forward way of proving this is to construct a Cauchy-sequence that does not have a limit point in \(X\). One example is
\[
f_n(x) = \begin{cases} 
-1 & x < 1/2 - 1/n, \\
(n(x - 1/2)) & 1/2 - 1/n \leq x \leq 1/2 + 1/n, \\
1 & x > 1/2 + 1/n.
\end{cases}
\]
We first prove that \((f_n)\) is Cauchy. Note that for any \(m, n,\) and \(x,\) we have \(|f_n(x) - f_m(x)| \leq 1.\) When \(m, n \geq N,\) we further have \(f_n(x) - f_m(x) = 0\) outside the interval \([1/2 - 1/N, 1/2 + 1/N],\) so
\[
||f_n - f_m|| = \int_{1/2-1/N}^{1/2+1/N} |f_n(x) - f_m(x)| \, dx \leq \int_{1/2-1/N}^{1/2+1/N} 1 \, dx = 2/N.
\]

We next prove that \((f_n)\) cannot converge to any element in \(X.\) Pick an arbitrary \(\varphi \in X.\) Assume temporarily that \(\varphi(1/2) \geq 0.\) Since \(\varphi\) is continuous, there exists a \(\delta > 0\) such that \(\varphi(x) \geq -1/2\) for \(x \in B_\delta(1/2).\) Pick an integer \(N > 2/\delta.\) Then, for \(n \geq N,\) we have \(f_n(x) = -1\) when \(x \in [1/2 - \delta, 1/2 - \delta/2],\) and so
\[
||f_n - \varphi|| \geq \int_{1/2-\delta}^{1/2-\delta/2} |f_n(x) - \varphi(x)| \, dx \geq \int_{1/2-\delta}^{1/2-\delta/2} 1/2 \, dx = \delta/4.
\]
If on the other hand \(\varphi(1/2) < 0,\) then pick \(\delta > 0\) such that \(\varphi(x) \leq 1/2\) on \([1/2, 1/2 + \delta]\) and proceed analogously.

**Remark 1:** Note that you cannot solve a problem like the one above by constructing a Cauchy sequence \((f_n)\) in \(X,\) point to a non-continuous function \(f,\) and claim that since \(f_n \text{ “converges to } f,\) \(X\) cannot be complete. Note that the metric is *not even defined* for functions outside of \(X.\)

**Remark 2:** Can you somehow add the limit points of Cauchy sequences in \(X\) and obtain a complete space \(\hat{X}?\) The answer is yes, you can do that for any metric space; the resulting space \(\hat{X}\) is called the “completion” of \(X\) and is (in a certain sense) unique. For the present example, \(X\) is the set of all (Lebesgue measurable) real-valued functions on \(I\) for which
\[
\int_0^1 |f(x)| \, dx < \infty,
\]
where the integral is what is called a “Lebesgue” integral. This space is denoted \(L^1(I).\) Strictly speaking, an element of \(L^1(I)\) is an equivalence class of functions that differ only on a set of Lebesgue measure zero. This roughly means that two functions \(f\) and \(g\) are considered identical if
\[
\int_0^1 |f(x) - g(x)| \, dx = 0.
\]