Problem 1: (16p) No motivations required for these problems. 4p each.

(a) Let $X$ be a set, and let $T$ denote a topology on $X$. Define what it means for $T$ to satisfy the Hausdorff property.

(b) Let $X$ denote a Banach space. Mark the following statements as true/false:

<table>
<thead>
<tr>
<th>Statement</th>
<th>TRUE</th>
<th>FALSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $(T_n)_{n=1}^\infty$ is a sequence in $\mathcal{B}(X)$ of compact operators that converges in norm to an operator $T$, then $T$ is necessarily compact.</td>
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<tr>
<td>Let $S,T \in \mathcal{B}(X)$. If $S$ is compact, then $ST$ is compact.</td>
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<td>Let $S,T \in \mathcal{B}(X)$. If $S$ and $T$ are both compact, then $S+T$ is compact.</td>
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(c) Set $I = [0,1]$ and $X = C(I)$. (We use the standard norm on $X$.) Define the subset $A = \{u \in X : u$ is continuously differentiable and $\|u'\| \leq 1\}$.

Describe the closure $\bar{A}$ of $A$:

Is $\bar{A}$ a compact set (yes/no)?

(d) Set $H = L^2([-1,1])$, and define $T \in \mathcal{B}(H)$ via $[Tu](x) = 2u(-x)$. Let $S \in \mathcal{B}(H)$ be an operator for which you know that $\|S\| \leq c$, where $c$ is some positive number. Are there any values of $c$ for which you can say for sure that the operator $T - S$ has closed range?

Answer:
**Problem 2:** (16p) Let $H$ denote a Hilbert space. Prove that for every element $\varphi \in H^*$, there exists a unique $y \in H$ such that

$$\varphi(x) = (y, x), \quad \forall x \in H.$$ 

**Problem 3:** (16p) Set $I = [0, \pi]$ and let $H$ denote the Hilbert space $H = L^2(I)$ with the usual norm. Define $f, g, h \in H$ via

$$f(x) = \sin(x), \quad g(x) = \sin(3x), \quad h(x) = x.$$ 

Set $N = \text{Span}\{f, g\}$, and $M = N^\perp$. Evaluate

$$d = \inf_{u \in M} \|h - u\|.$$ 

In the event that you make any computational errors, your score on this problem will depend strongly on whether you clearly describe the process you use to determine $d$.

**Problem 4:** (16p) Set $I = [0, 2]$, set $X = C(I)$, and let $k$ be a continuous function on $I \times I$. Consider the operator $T \in \mathcal{B}(X)$ defined by

$$[Tu](x) = \int_0^2 k(x, y) u(y) dy, \quad x \in I.$$ 

(a) State the Arzelá-Ascoli theorem.

(b) Prove that the operator $T$ is compact.

**Problem 5:** (16p) Let $X$ denote the space of all continuous functions on $\mathbb{R}$ that are periodic with period 1. In other words, if $u \in X$, then

$$u(x) = u(x + 1), \quad \forall x \in \mathbb{R}.$$ 

We equip $X$ with the norm

$$\|u\| = \sup_{x \in [0,1]} |u(x)|.$$ 

Observe that a function $u$ in $X$ is uniquely defined by its values on the interval $I = [0, 1]$ (or on $[0, 1)$, for that matter, since $u(0) = u(1)$). Define for $n = 1, 2, 3, \ldots$ the operators

$$[T_n u](x) = u(x - 1/n).$$ 

(a) (6p) Does $(T_n)_{n=1}^\infty$ converge strongly? Please motivate your answer carefully.

(b) (6p) Does $(T_n)_{n=1}^\infty$ converge in norm? Please motivate your answer carefully.

(c) (4p) Do your answers change if $X$ is instead equipped with the norm $\|u\| = \int_0^1 |u(x)| \, dx$?