Let $\mathcal{X}$ be a Banach space and consider the space $\mathcal{X}^*$. We are by now familiar with two types of convergence in $\mathcal{X}^*$:

1. $\varphi_n \to \varphi$ in norm if $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$.
2. $\varphi_n \to \varphi$ weakly if $F(\varphi_n) \to F(\varphi)$ for all $F \in \mathcal{X}^{**}$.

We next define a third mode of convergence:

3. $\varphi_n \to \varphi$ weak-$*$ if $\varphi_n(x) \to \varphi(x)$ for all $x \in \mathcal{X}$.

Since $\mathcal{X} \subseteq \mathcal{X}^*$, the weak-$*$ topology is a weaker (or equivalent) topology to the weak topology.

If $\mathcal{X}$ is reflexive, $\mathcal{X}^{**} = \mathcal{X}$, then the weak and the weak-$*$ topologies are the same.

The Hahn-Banach theorem implies that the weak-$*$ topology is Hausdorff.

The weak-$*$ topology is useful because the unit ball is typically compact in this topology. To be precise, set $S^* = \{\varphi \in \mathcal{X}^* : \|\varphi\| \leq 1\}$. Then $S^*$ is compact in the norm top $\Rightarrow \mathcal{X}$ is finite dim. $S^*$ is compact in the weak top $\Rightarrow \mathcal{X}$ is reflexive. $S^*$ is always compact in the weak-$*$ top!
Alaoglu

Let $I$ be a NLS.

Let $S^*$ denote the closed unit ball in $I^*$.

Then $S^*$ is a compact Hausdorff space in the weak-$*$ topology.

Theorem

Let $I$ be a Banach space and let $S$ denote its closed unit ball. Then:

$S$ is compact in the weak-$*$ topology $\iff I$ is reflexive.