Problem 1: Let $(X, d)$ be a metric space.

(a) Define what it means for a subset $\Omega$ of $X$ to be compact. (2p)

(b) Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\Omega$. Suppose that there exists an $\varepsilon > 0$, such that if $n \neq m$, then $d(x_n, x_m) \geq \varepsilon$. Prove directly from the definition given in (a) that $\Omega$ cannot be compact. (2p)

(*) If you have time left, this problem could earn you 1p extra: Give a definition of compactness that is equivalent to, but different from, the one you gave in (a), and redo (b) using this definition.

Problem 2: State the Grönwall inequality. (2p)

Problem 3: Consider the equation

\[ u(x) + \int_0^x u(y)^3 \, dy = 1. \]

Prove that for some positive $\delta$, equation (1) has a unique solution in $C_b([0, \delta])$. (5p)

Hint: You may want to work with a bounded subspace of $C_b([0, \delta])$, rather than the space itself.

Problem 4: Set $I = (0,1)$, $X = C_b(I)$, and

$\Omega = \{ f \in X : f \text{ has compact support in } I \text{ and } ||f|| \leq 1 \}.$

(a) Prove that $\Omega$ is not closed in $X$. (2p)

(b) Describe $\bar{\Omega}$, the closure of $\Omega$ in $X$. (1p)

(c) Is $\bar{\Omega}$ a compact set in $X$? Prove that your answer is correct. (3p)

Problem 5: Let $f$ denote a fixed function in $C_b(\mathbb{R})$. Set $\Omega = \{ f_n \}_{n=1}^{\infty}$ where $f_n(x) = f(x - n)$. Is $\Omega$ necessarily equicontinuous? Motivate your answer. (3p)