Homework set 3 — APPM5440

From the textbook: 1.17, 1.18, 1.20, 1.21, 1.22.

Note that you can find solutions of many textbook problems at:
http://www.math.ucdavis.edu/~bxn/

Problem: Suppose that \((x_n)_{n=1}^\infty\) and \((y_n)_{n=1}^\infty\) are Cauchy sequences in a metric space. Prove that \((d(x_n, y_n))_{n=1}^\infty\) converges.

Problem: In the proof that every metric space has a completion, we did not prove that the function \(\tilde{d}\) is indeed a metric. Verify that this is the case.

Optional problem: In the proof that every metric space has a completion, the technique we used to prove that \(\tilde{X}\) is complete is called the “Cantor diagonal argument”. This is a standard technique. Try to use it to prove that the real numbers are not countable. Hint: Assume that there exists an enumeration \((r^{(n)})_{n=1}^\infty\) of all real numbers in the interval \((0, 1)\). Suppose that each \(r^{(n)}\) has a binary number expansion
\n\[ r^{(n)} = 0.b_1^{(n)} b_2^{(n)} b_3^{(n)} \ldots \]
\n(so that each \(b_j^{(n)}\) is either 0 or 1) and use the “diagonal” technique to construct a real number that is not in the sequence. (There is a full solution in the Wikipedia article on Cantor’s diagonal argument.)