Section exam 2 Numerical Analysis: Linear Algebra
Oct. 15, 2026. 75 minutes.

Instructions:

- ***This is a closed-books exam.*** No books or calculators are allowed. However, you may bring one hand-written, single-sided formula sheet.
- Try to answer the questions within the space given. If you need extra pages, attach them and indicate clearly which question they address.
- If a question does not specifically ask for a justification, the correct answer alone earns full credit. You are welcome to include a brief explanation, which may help you receive partial credit (except for Question 1, where only the answer will be graded).
- Use your time strategically. You may wish to save problems 2(c), 4(d), and 5 for last – they do not provide many points in relation to the amount of work required.
- Unless otherwise indicated, $\|\mathbf{x}\|$ denotes the Euclidean norm, $\|\mathbf{A}\|$ the corresponding operator norm, and $\|\mathbf{A}\|_{\mathrm{F}}$ the Frobenius norm. The notation $\mathbf{A}^*$ denotes conjugate transpose. We use $\kappa(\mathbf{A})$ to denote the Euclidean norm matrix condition number, so that $\kappa(\mathbf{A}) = \sigma_{\max}(\mathbf{A})/\sigma_{\min}(\mathbf{A})$.

Name: _____

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Question	Max points	Scored points
1	30	
2	15	
3	20	
4	20	
5	15	
Total	**100**	

Question 1: (30p) For every part of this question, give only your answers. Derivations or explanations are not required, and will not be graded.

(a) (10p) Consider a hypothetical 26-bit binary floating-point format, with one sign bit, five exponent bits, and twenty stored fraction bits. Assume the format uses the same exponent conventions as IEEE floating-point arithmetic.

(i) (3p) Specify, as a power of two, the unit roundoff:

$$\epsilon_{\text{mach}} \approx$$

(ii) (4p) Specify, as a power of two, the magnitude of the smallest and largest positive numbers that you can accurately represent in this format:

$$f_{\min} \approx \qquad\qquad f_{\max} \approx$$

(iii) (3p) Approximately how many decimal significant digits can you represent?

Note: Answers to (i) and (ii) that are accurate within a factor of 2 will be graded as correct. You do not need to worry about exactly how rounding is done, etc. (And do not worry about "subnormal" numbers or anything like that.)

(b) (5p) Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ be a matrix. You know that

$$\sup_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| = 4, \qquad \inf_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\| = 1, \qquad \|\mathbf{A}\|_{\text{F}} = \sqrt{21}.$$

Specify the three singular values of $\mathbf{A}$, in decreasing order:

$$\sigma_1(\mathbf{A}) = \qquad \sigma_2(\mathbf{A}) = \qquad \sigma_3(\mathbf{A}) =$$

(c) (15p) The following matrices satisfy $\mathbf{A} = \mathbf{Q}\mathbf{R}$, where $\mathbf{Q}$ is orthogonal and $\mathbf{R}$ is upper triangular. Each cross represents one missing scalar. Specify these missing entries.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & \times & 2 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & \times \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 2 & 2 & 0 & \times \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & \times \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

$$\mathbf{A}(4,3) = \qquad \mathbf{Q}(4,4) = \qquad \mathbf{R}(1,4) = \qquad \mathbf{R}(3,4) =$$

Question 2: (15p) Consider two matrices $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{4\times 3}$ with reduced QR factorizations $\mathbf{A}_1 = \mathbf{Q}\mathbf{R}_1$ and $\mathbf{A}_2 = \mathbf{Q}\mathbf{R}_2$, where

$$\mathbf{Q} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

The columns of $\mathbf{Q}$ are orthonormal. The matrices and their triangular factors are

$$\mathbf{A}_1 = \begin{bmatrix} 3 & 2 & 1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \\ 3 & -2 & -1 \end{bmatrix}, \qquad \mathbf{R}_1 = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{A}_2 = \frac{1}{2}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & -2 \\ 1 & 0 & -2 \end{bmatrix}, \qquad \mathbf{R}_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

No proofs or derivations are required.

(a) (6p) Specify the singular values of $\mathbf{A}_1$, in decreasing order.

(b) (4p) Specify $\kappa(\mathbf{A}_1)$ and $\kappa(\mathbf{A}_1^*\mathbf{A}_1)$.

(c) (5p) Specify the singular values of $\mathbf{A}_2$, in decreasing order.

Question 3: (20p) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be nonsingular, with SVD

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*, \qquad \mathbf{\Sigma} = \operatorname{diag}(\sigma_1, \ldots, \sigma_m), \quad \sigma_1 \geq \cdots \geq \sigma_m > 0.$$

Write $\mathbf{u}_j$ and $\mathbf{v}_j$ for the columns of $\mathbf{U}$ and $\mathbf{V}$. Consider the self-adjoint block matrix

$$\mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^* \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \in \mathbb{C}^{2m\times 2m}.$$

(a) (10p) For each $j = 1, \ldots, m$, find two eigenvectors of $\mathbf{H}$ in terms of $\mathbf{u}_j$ and $\mathbf{v}_j$, and verify their eigenvalue equations.

(b) (6p) List all $2m$ eigenvalues of $\mathbf{H}$, including multiplicities.

(c) (4p) Express $\|\mathbf{H}\|$ and $\|\mathbf{H}\|_{\mathrm{F}}$ in terms of the singular values of $\mathbf{A}$.

Question 4: (20p) Set $t = 10^{-6}$. Consider an equation $\mathbf{Ax} = \mathbf{b}$ with a proposed approximate solution $\widehat{\mathbf{x}}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & t \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ t \end{bmatrix}, \qquad \widehat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(a) (5p) Find the exact solution $\mathbf{x}_\star = \mathbf{A}^{-1}\mathbf{b}$.

(b) (5p) Compute the relative residual $\|\mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}\|/\|\mathbf{b}\|$.

(c) (5p) Compute the relative forward error $\|\mathbf{x}_\star - \widehat{\mathbf{x}}\|/\|\mathbf{x}_\star\|$.

(d) (5p) Now regard t as a variable small positive real number. Prove that as $t \searrow 0$, it is the case that $t\kappa(\mathbf{A}) \to C$ for some constant C. (So that $\kappa(A) \approx C/t$ for small t.) Specify C. *Hint: Part (d) requires a little bit of work, and you may want to save it for last, since it involves only 5p.*

Question 5: (15p) Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be nonsingular, and let $\mathbf{B}$ be an approximate inverse that satisfies

$$\|\mathbf{B} - \mathbf{A}^{-1}\| \leq 10^{-6} \|\mathbf{A}^{-1}\|.$$

Given a vector $\mathbf{b}$, we attempt to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$, through an iteration that starts with $\mathbf{x}_1 = \mathbf{B}\mathbf{b}$, and then proceeds through

$$\mathbf{r}_k = \mathbf{b} - \mathbf{A}\mathbf{x}_k, \qquad \mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{B}\mathbf{r}_k, \qquad k = 1, 2, 3, \ldots$$

For this question, ignore all rounding errors beyond the initial inaccuracy of $\mathbf{B}$. We observe the following relative residual norms:

k	1	2	3	4
$\|\mathbf{r}_k\|/\|\mathbf{b}\|$	10^{-3}	10^{-5}	10^{-7}	10^{-9}

(a) (4p) Show that $\mathbf{r}_{k+1} = \mathbf{M}\mathbf{r}_k$ for some matrix $\mathbf{M}$. Specify $\mathbf{M}$.

(b) (7p) What can you tell about the condition number of $\mathbf{A}$, based on the observed norms $\|\mathbf{r}_k\|$? Explain your reasoning.

(c) (4p) If the observed factor-of-100 reduction continues, for what k would you expect $\|\mathbf{r}_k\|/\|\mathbf{b}\|$ to be less than 10^{-14}? At this point, what would your expectation be in regards to $\|\mathbf{x}_\star - \mathbf{x}_k\|/\|\mathbf{x}_\star\|$?