

Section exam 1 Numerical Analysis: Linear Algebra — Solutions

Sep. 24, 2026. 75 minutes.

Instructions:

- ***This is a closed books exam.*** No books or calculators are allowed. However, you are allowed to bring one hand written, single sided formula sheet.
- Try to answer the questions within the space given. If you must use more space, you *can* attach extra pages, but try to avoid it. If you do need extra space, then please mark this very clearly on the exam.
- If a question does not specifically ask for a motivation, then the correct answer alone will give full points. You are welcome to enter a brief explanation of how you arrived at the answer — this might yield partial credit in case the given answer is incorrect.
- Use your time strategically. Question 1 should be fast. Questions 2 – 4 require some computation or reasoning. Question 5 is intended to be the most challenging question, but is worth only 15 points, so it may be wise to save it for last.
- In this exam, the default is that for a vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ denotes the Euclidean norm. For a matrix $\mathbf{A}$, the notation $\|\mathbf{A}\|$ refers to the operator norm with respect to the Euclidean vector norm, and $\|\mathbf{A}\|_F$ refers to the Frobenius norm. If $\mathbf{A}$ is a matrix, then $\mathbf{A}^*$ refers to the complex conjugate of the transpose of $\mathbf{A}$ (or merely the transpose when $\mathbf{A}$ is real).

Question 1: (25p) Give brief answers.

- (a) (6p) Let $\mathbf{Q} \in \mathbb{C}^{m \times n}$, where $m \geq n$, have orthonormal columns. Specify

$$\|\mathbf{Q}\| \quad \text{and} \quad \|\mathbf{Q}\|_F.$$

Since $\|\mathbf{Q}\mathbf{x}\| = \|\mathbf{x}\|$ for every vector $\mathbf{x}$, we have $\|\mathbf{Q}\| = 1$.

To evaluate the Frobenius norm, set $\mathbf{Q} = [\mathbf{q}_1, \dots, \mathbf{q}_n]$. Then $\|\mathbf{Q}\|_F^2 = \sum_{j=1}^n \|\mathbf{q}_j\|^2 = \sum_{j=1}^n 1 = n$.
Hence $\|\mathbf{Q}\|_F = \sqrt{n}$.

- (b) (6p) Let $\mathbf{P} \neq \mathbf{0}$ be a projection, so that $\mathbf{P}^2 = \mathbf{P}$.

Is it possible for $\|\mathbf{P}\| < 1$? Motivate your answer or give an example.

Is it possible for $\|\mathbf{P}\| > 1$? Motivate your answer or give an example.

Since $\mathbf{P} \neq \mathbf{0}$, there is a nonzero $\mathbf{x} \in \text{ran}(\mathbf{P})$. For such a vector, $\mathbf{P}\mathbf{x} = \mathbf{x}$, and therefore

$$\|\mathbf{P}\| \geq \frac{\|\mathbf{P}\mathbf{x}\|}{\|\mathbf{x}\|} = 1.$$

Therefore, $\|\mathbf{P}\| \geq 1$.

It is possible that $\|\mathbf{P}\| > 1$. For example,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

satisfies $\mathbf{P}^2 = \mathbf{P}$, while

$$\|\mathbf{P}\| = \sqrt{2} > 1.$$

(For a nonzero orthogonal projection, $\|\mathbf{P}\| = 1$.)

- (c) (6p) Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ have rank r . Specify

$$\dim(\text{col}(\mathbf{A})) \quad \text{and} \quad \dim(\text{null}(\mathbf{A})).$$

By the definition of rank and the rank-nullity theorem,

$$\boxed{\dim(\text{col}(\mathbf{A})) = r}, \quad \boxed{\dim(\text{null}(\mathbf{A})) = n - r}.$$

- (d) (7p) Let $\mathbf{A} \in \mathbb{C}^{m \times n}$, $m \geq n$, have full column rank, and suppose that

$$\mathbf{A} = \mathbf{Q}\mathbf{R}$$

is a reduced (“economy size”) QR factorization. Give a formula for the orthogonal projector onto $\text{col}(\mathbf{A})$. What are the dimensions of this projector?

The columns of $\mathbf{Q}$ form an orthonormal basis for $\text{col}(\mathbf{A})$, so

$$\boxed{\mathbf{P} = \mathbf{Q}\mathbf{Q}^*}.$$

Since $\mathbf{Q}$ is $m \times n$, the projector $\mathbf{P}$ is

$$\boxed{m \times m}.$$

Question 2: (20p) Let $\mathbf{A} \in \mathbb{R}^{4 \times 3}$ have the singular value decomposition

$$\mathbf{A} = \mathbf{U} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}^*,$$

where

$$\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4], \quad \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

are orthogonal matrices. Let

$$\mathbf{b} = 8\mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_4.$$

(a) (7p) Find the least squares solution

$$\mathbf{x}_* = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{Ax} - \mathbf{b}\|.$$

Express your answer in terms of the vectors $\mathbf{v}_j$.

Let $\mathbf{D}$ denote the 3×3 matrix with diagonal entries $[4, 2, 1]$ and set $\hat{\mathbf{U}} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$, so that $\mathbf{A} = \hat{\mathbf{U}}\mathbf{D}\mathbf{V}^*$. Then

$$\mathbf{x}_* = \mathbf{V}\mathbf{D}^{-1}\hat{\mathbf{U}}^*\mathbf{b} = \mathbf{V}\mathbf{D}^{-1} \begin{bmatrix} 8 \\ 2 \\ 0 \end{bmatrix} = \mathbf{V} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \boxed{2\mathbf{v}_1 + \mathbf{v}_2}.$$

(b) (5p) Specify $\|\mathbf{r}\|$ where $\mathbf{r} = \mathbf{b} - \mathbf{Ax}_*$ is the residual.

Since

$$\mathbf{Ax}_* = 8\mathbf{u}_1 + 2\mathbf{u}_2,$$

we have

$$\mathbf{r} = 3\mathbf{u}_4, \quad \boxed{\|\mathbf{r}\| = 3}.$$

(c) (4p) Specify a matrix $\mathbf{A}_1$ that is an optimal rank-one approximation to $\mathbf{A}$ in the operator norm. Your answer may involve the vectors $\{\mathbf{u}_i\}_{i=1}^4$ and $\{\mathbf{v}_i\}_{i=1}^3$.

By the Eckart–Young theorem, truncate the SVD after its first term:

$$\boxed{\mathbf{A}_1 = 4\mathbf{u}_1\mathbf{v}_1^*}.$$

(d) (4p) Specify

$$\|\mathbf{A} - \mathbf{A}_1\| \quad \text{and} \quad \|\mathbf{A} - \mathbf{A}_1\|_F.$$

The remaining singular values are 2 and 1. Therefore

$$\boxed{\|\mathbf{A} - \mathbf{A}_1\| = 2}, \quad \boxed{\|\mathbf{A} - \mathbf{A}_1\|_F = \sqrt{2^2 + 1^2} = \sqrt{5}}.$$

Note: Any matrix of the form $\mathbf{A}_1 = \mu\mathbf{u}_1\mathbf{v}_1^*$ with $\mu \in [2, 6]$ is an optimal rank-one approximation in the spectral norm. For this choice, we have

$$\|\mathbf{A} - \mathbf{A}_1\| = 2, \quad \|\mathbf{A} - \mathbf{A}_1\|_F = \sqrt{(\mu - 4)^2 + 2^2 + 1^2}.$$

Question 3: (20p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Compute a reduced QR factorization

$$\mathbf{A} = \mathbf{Q}\mathbf{R}$$

using the *modified Gram-Schmidt process*. Use the convention that the diagonal entries of $\mathbf{R}$ should be positive. Give both the 3×2 matrix $\mathbf{Q}$ and the 2×2 matrix $\mathbf{R}$ explicitly.

Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$r_{11} = \|\mathbf{a}_1\| = \sqrt{2}, \quad \mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Next,

$$r_{12} = \mathbf{q}_1^* \mathbf{a}_2 = \frac{1}{\sqrt{2}},$$

and hence

$$\mathbf{w}_2 = \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

Thus

$$r_{22} = \|\mathbf{w}_2\| = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}, \quad \mathbf{q}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Therefore

$$\boxed{\mathbf{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix}}, \quad \boxed{\mathbf{R} = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{bmatrix}}.$$

Question 4: (20p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

We seek a Householder reflector $\mathbf{F}$ such that

$$\mathbf{F} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}.$$

- (a) (6p) Find a Householder vector $\mathbf{v}$ that accomplishes this transformation. You do *not* need to normalize $\mathbf{v}$.

Let

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{y} = -\text{sign}(x_1)\|\mathbf{x}\|\mathbf{e}_1 = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}.$$

A normal vector to the reflecting plane is proportional to $\mathbf{x} - \mathbf{y} = [8, 4, 0]^*$. Thus one convenient choice is

$$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

- (b) (7p) Compute the corresponding reflector

$$\mathbf{F} = \mathbf{I} - 2 \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}.$$

For $\mathbf{v} = [2, 1, 0]^*$,

$$\mathbf{v}^*\mathbf{v} = 5, \quad \mathbf{v}\mathbf{v}^* = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore

$$\mathbf{F} = \begin{bmatrix} -3/5 & -4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (c) (7p) Compute $\mathbf{FA}$.

Multiplying the two columns of $\mathbf{A}$ by $\mathbf{F}$ gives

$$\mathbf{F} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{F} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/5 \\ -4/5 \\ 1 \end{bmatrix}.$$

Hence

$$\mathbf{FA} = \begin{bmatrix} -5 & -3/5 \\ 0 & -4/5 \\ 0 & 1 \end{bmatrix}.$$

Question 5: (15p) Let

$$\mathbf{A} \in \mathbb{C}^{m \times n}, \quad \text{rank}(\mathbf{A}) = r < n.$$

Suppose that we have constructed a unitary matrix $\mathbf{Q} \in \mathbb{C}^{m \times m}$ such that

$$\mathbf{Q}^* \mathbf{A} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}, \quad (1)$$

where $\mathbf{B} \in \mathbb{C}^{r \times n}$ has full row rank.

Let $\mathbf{B}^*$ have the reduced QR factorization

$$\mathbf{B}^* = \mathbf{V} \mathbf{T}, \quad (2)$$

where $\mathbf{V} \in \mathbb{C}^{n \times r}$ has orthonormal columns and $\mathbf{T} \in \mathbb{C}^{r \times r}$ is nonsingular and upper triangular. Let

$$\mathbf{U} = \mathbf{Q}(:, 1 : r).$$

(a) (4p) Show that

$$\mathbf{A} = \mathbf{U} \mathbf{T}^* \mathbf{V}^*. \quad (3)$$

Multiplying (1) by $\mathbf{Q}$, we get

$$\mathbf{A} = \mathbf{Q} \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} = [\mathbf{U}, \mathbf{Q}(:, (r+1) : m)] \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} = \mathbf{U} \mathbf{B}. \quad (4)$$

Taking the transpose of (2), we get $\mathbf{B} = \mathbf{T}^* \mathbf{V}^*$. Inserting this to (4) yields

$$\mathbf{A} = \mathbf{U} \mathbf{T}^* \mathbf{V}^*. \quad (5)$$

(b) (6p) Let $\mathbf{b} \in \mathbb{C}^m$, and define $\mathbf{x}_\star = \mathbf{V}(\mathbf{T}^*)^{-1} \mathbf{U}^* \mathbf{b}$. Prove that $\|\mathbf{A} \mathbf{x}_\star - \mathbf{b}\| = \inf_{\mathbf{y} \in \mathbb{C}^n} \|\mathbf{A} \mathbf{y} - \mathbf{b}\|$.

From (3) we find that

$$\mathbf{A} \mathbf{x}_\star = \mathbf{U} \mathbf{T}^* \mathbf{V}^* \mathbf{V} (\mathbf{T}^*)^{-1} \mathbf{U}^* \mathbf{b} = \mathbf{U} \mathbf{T}^* (\mathbf{T}^*)^{-1} \mathbf{U}^* \mathbf{b} = \mathbf{U} \mathbf{U}^* \mathbf{b}.$$

The columns of $\mathbf{U}$ form an orthonormal basis for $\text{col}(\mathbf{A})$, so $\mathbf{U} \mathbf{U}^* \mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ onto $\text{col}(\mathbf{A})$. Hence $\mathbf{A} \mathbf{x}_\star$ is the point in $\text{col}(\mathbf{A})$ closest to $\mathbf{b}$. Therefore

$$\|\mathbf{A} \mathbf{x}_\star - \mathbf{b}\| = \inf_{\mathbf{y} \in \mathbb{C}^n} \|\mathbf{A} \mathbf{y} - \mathbf{b}\|.$$

(c) (5p) Prove that $\mathbf{x}_\star$ is the unique least squares solution having minimum Euclidean norm. You may find it useful to characterize $\text{null}(\mathbf{A})$ in terms of the columns of $\mathbf{V}$.

Let $\mathbf{x}$ be another least squares solution.

Let us first show that $\mathbf{x}_\star \in \text{null}(\mathbf{A})^\perp$. To see this, observe that

$$\mathbf{A} \mathbf{z} = \mathbf{0} \iff \mathbf{V}^* \mathbf{z} = \mathbf{0}$$

since $\mathbf{U}$ is orthonormal and $\mathbf{T}$ is non-singular. It follows that $\text{null}(\mathbf{A}) = \text{null}(\mathbf{V}^*) = \text{col}(\mathbf{V})^\perp$ and since $\mathbf{x}_\star \in \text{col}(\mathbf{V})$, the claim follows.

Next we show that $\mathbf{x}_\star - \mathbf{x} \in \text{null}(\mathbf{A})$. To see why, observe that it must be the case that $\mathbf{A} \mathbf{x} = \mathbf{U} \mathbf{U}^* \mathbf{b}$, so

$$\mathbf{A}(\mathbf{x}_\star - \mathbf{x}) = \mathbf{A} \mathbf{x}_\star - \mathbf{A} \mathbf{x} = \mathbf{U} \mathbf{U}^* \mathbf{b} - \mathbf{U} \mathbf{U}^* \mathbf{b} = \mathbf{0}.$$

Combining our two findings, we see that $\mathbf{x}_\star$ and $\mathbf{x}_\star - \mathbf{x}$ are orthogonal, and so

$$\|\mathbf{x}\|^2 = \|\mathbf{x}_\star - (\mathbf{x}_\star - \mathbf{x})\|^2 = \|\mathbf{x}_\star\|^2 + \|\mathbf{x}_\star - \mathbf{x}\|^2.$$

It follows that $\mathbf{x}_\star$ must be the unique minimal norm solution.