

Section exam 1 Numerical Analysis: Linear Algebra

Sep. 24, 2026. 75 minutes.

Instructions:

- ***This is a closed books exam.*** No books or calculators are allowed. However, you are allowed to bring one hand written, single sided formula sheet.
- Try to answer the questions within the space given. If you must use more space, you *can* attach extra pages, but try to avoid it. If you do need extra space, then please mark this very clearly on the exam.
- If a question does not specifically ask for a motivation, then the correct answer alone will give full points. You are welcome to enter a brief explanation of how you arrived at the answer — this might yield partial credit in case the given answer is incorrect.
- Use your time strategically. Question 1 should be fast. Questions 2 – 4 require some computation or reasoning. Question 5 is intended to be the most challenging question, but is worth only 15 points, so it may be wise to save it for last.
- In this exam, the default is that for a vector $\mathbf{x}$, the notation $\|\mathbf{x}\|$ denotes the Euclidean norm. For a matrix $\mathbf{A}$, the notation $\|\mathbf{A}\|$ refers to the operator norm with respect to the Euclidean vector norm, and $\|\mathbf{A}\|_F$ refers to the Frobenius norm. If $\mathbf{A}$ is a matrix, then $\mathbf{A}^*$ refers to the complex conjugate of the transpose of $\mathbf{A}$ (or merely the transpose when $\mathbf{A}$ is real).

Name: _____

Section: _____

Question	Max points	Scored points
1	25	
2	20	
3	20	
4	20	
5	15	
Total:	100	

Question 1: (25p) Give brief answers.

(a) (6p) Let $\mathbf{Q} \in \mathbb{C}^{m \times n}$, where $m \geq n$, have orthonormal columns. Specify

$$\|\mathbf{Q}\| \quad \text{and} \quad \|\mathbf{Q}\|_{\text{F}}.$$

(b) (6p) Let $\mathbf{P} \neq \mathbf{0}$ be a projection, so that $\mathbf{P}^2 = \mathbf{P}$.

Is it possible for $\|\mathbf{P}\| < 1$? Motivate your answer or give an example.

Is it possible for $\|\mathbf{P}\| > 1$? Motivate your answer or give an example.

(c) (6p) Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ have rank r . Specify

$$\dim(\text{col}(\mathbf{A})) \quad \text{and} \quad \dim(\text{null}(\mathbf{A})).$$

(d) (7p) Let $\mathbf{A} \in \mathbb{C}^{m \times n}$, $m \geq n$, have full column rank, and suppose that

$$\mathbf{A} = \mathbf{Q}\mathbf{R}$$

is a reduced (“economy size”) QR factorization. Give a formula for the orthogonal projector onto $\text{col}(\mathbf{A})$. What are the dimensions of this projector?

Question 2: (20p) Let $\mathbf{A} \in \mathbb{R}^{4 \times 3}$ have the singular value decomposition

$$\mathbf{A} = \mathbf{U} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}^*,$$

where

$$\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4], \quad \mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

are orthogonal matrices. Let

$$\mathbf{b} = 8\mathbf{u}_1 + 2\mathbf{u}_2 + 3\mathbf{u}_4.$$

(a) (7p) Find the least squares solution

$$\mathbf{x}_\star = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{Ax} - \mathbf{b}\|.$$

Express your answer in terms of the vectors $\mathbf{v}_j$.

(b) (5p) Specify $\|\mathbf{r}\|$ where $\mathbf{r} = \mathbf{b} - \mathbf{Ax}_\star$ is the residual.

(c) (4p) Specify a matrix $\mathbf{A}_1$ that is an optimal rank-one approximation to $\mathbf{A}$ in the operator norm. Your answer may involve the vectors $\{\mathbf{u}_i\}_{i=1}^4$ and $\{\mathbf{v}_i\}_{i=1}^3$.

(d) (4p) Specify

$$\|\mathbf{A} - \mathbf{A}_1\| \quad \text{and} \quad \|\mathbf{A} - \mathbf{A}_1\|_F.$$

Question 3: (20p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Compute a reduced QR factorization

$$\mathbf{A} = \mathbf{QR}$$

using the *modified Gram–Schmidt process*. Use the convention that the diagonal entries of $\mathbf{R}$ should be positive. Give both the 3×2 matrix $\mathbf{Q}$ and the 2×2 matrix $\mathbf{R}$ explicitly.

Question 4: (20p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

We seek a Householder reflector $\mathbf{F}$ such that

$$\mathbf{F} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 0 \end{bmatrix}.$$

- (a) (6p) Find a Householder vector $\mathbf{v}$ that accomplishes this transformation. You do *not* need to normalize $\mathbf{v}$.

- (b) (7p) Compute the corresponding reflector

$$\mathbf{F} = \mathbf{I} - 2 \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}.$$

- (c) (7p) Compute $\mathbf{FA}$.

Question 5: (15p) Let

$$\mathbf{A} \in \mathbb{C}^{m \times n}, \quad \text{rank}(\mathbf{A}) = r < n.$$

Suppose that we have constructed a unitary matrix $\mathbf{Q} \in \mathbb{C}^{m \times m}$ such that

$$\mathbf{Q}^* \mathbf{A} = \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{B} \in \mathbb{C}^{r \times n}$ has full row rank. Let $\mathbf{B}^*$ have the reduced QR factorization

$$\mathbf{B}^* = \mathbf{V} \mathbf{T},$$

where $\mathbf{V} \in \mathbb{C}^{n \times r}$ has orthonormal columns and $\mathbf{T} \in \mathbb{C}^{r \times r}$ is nonsingular and upper triangular. Let

$$\mathbf{U} = \mathbf{Q}(:, 1 : r).$$

(a) (4p) Show that

$$\mathbf{A} = \mathbf{U} \mathbf{T}^* \mathbf{V}^*.$$

(b) (6p) Let $\mathbf{b} \in \mathbb{C}^m$, and define $\mathbf{x}_\star = \mathbf{V}(\mathbf{T}^*)^{-1} \mathbf{U}^* \mathbf{b}$. Prove that $\|\mathbf{A} \mathbf{x}_\star - \mathbf{b}\| = \inf_{\mathbf{y} \in \mathbb{C}^n} \|\mathbf{A} \mathbf{y} - \mathbf{b}\|$.

Hint: Compute $\mathbf{A} \mathbf{x}_\star$ and interpret the result geometrically.

(c) (5p) Prove that $\mathbf{x}_\star$ is the unique least squares solution having minimum Euclidean norm. You may find it useful to characterize $\text{null}(\mathbf{A})$ in terms of the columns of $\mathbf{V}$.