Section exam 2 for M341: Linear Algebra and Matrix Theory

Thursday, March 28, 2024. 75 minutes exam time. *Closed books. No notes.*

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	Question 1 (25 max)	Question 2 (20 max)	Question 3 (15 max)	Question 4 (20 max)	Question 5 (15 max)	Question 6 (5 max)	Total (100 max)
Score:							

Question 1: (25p) In this question, we as usual let $\mathbf{X}^{\mathrm{T}}$ denote the *transpose* of a matrix $\mathbf{X}$. No motivation is required for these problems.

(a) (5p) Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 0 & -1 \\ 4 & 2 & 7 \\ 2 & 3 & 7 \end{bmatrix}$ and the vector $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. In answering this question, you may use that $\mathbf{A}$ is invertible, and that $\mathbf{A}^{-1} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix}$.

Specify the solution to the linear system $\mathbf{AX} = \mathbf{B}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ -10 \end{bmatrix}$$

(b) (5p) With $\mathbf{A}$ and $\mathbf{B}$ as in (a), specify the solution to the linear system $\mathbf{A}^{\mathrm{T}}\mathbf{Y} = \mathbf{B}$:

$$\mathbf{Y} = \left(\mathbf{A}^{\mathrm{T}}\right)^{-1}\mathbf{B} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} 7 & 14 & -8 \\ 3 & 5 & -3 \\ -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ -4 \end{bmatrix}$$

(c) (5p) Evaluate the following determinant: $\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}\right) = 1 \cdot 5 \cdot 8 \cdot 10 = 400$

(d) (5p) Let $\mathbf{A}$ be a 3×3 matrix such that $\det(\mathbf{A}) = 3$. Complete the equation: $\det(2\mathbf{A}) = 2^3 \cdot 3 = 24$

(e) (5p) In this problem, $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of the same dimensions. Circle the statements that are necessarily true.

(i) $\det(\mathbf{A}+\mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$. FALSE

(ii) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$. TRUE

(iii) Every matrix has at least one real eigenvalue. FALSE

(iv) If $\mathbf{X}$ and $\mathbf{Y}$ are two eigenvectors of $\mathbf{A}$, then $\mathbf{X}+\mathbf{Y}$ is also an eigenvector of $\mathbf{A}$. FALSE

(v) Suppose that W is a subspace of a vector space V, and that $\{\mathbf{v}_j\}_{j=1}^{n}$ is a collection of vectors in W. If $\mathbf{x}$ is a linear combination of the vectors $\{\mathbf{v}_j\}_{j=1}^{n}$, then $\mathbf{x} \in W$. TRUE

Question 2: (20p) Compute all eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

Please motivate your answers briefly.

Solution: First we evaluate the characteristic polynomial of $\mathbf{A}$:

$$p_{\mathbf{A}}(\lambda) = \det\bigl(\lambda\mathbf{I} - \mathbf{A}\bigr) = \det \begin{bmatrix} \lambda - 1 & 2 \\ -1 & \lambda - 4 \end{bmatrix} = (\lambda - 1)(\lambda - 4) + 2 = \lambda^2 - 5\lambda + 6$$

The eigenvalues are the solutions to $p_{\mathbf{A}}(\lambda) = 0$, so

$$\lambda = \frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{24}{4}} = \frac{5}{2} \pm \frac{1}{2}.$$

We see that the eigenvalues are 2 and 3.

Find eigenvectors for $\lambda = 2$: We seek the solutions to $(2\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We see that x_2 is free. Setting, e.g., $x_2 = 1$, we find the eigenvector

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Find eigenvectors for $\lambda = 3$: We seek the solutions to $(3\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We see that x_2 is free. Setting, e.g., $x_2 = 1$, we find the eigenvector

$$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Question 3: (15p) The matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 6$. Show your work when answering (a) and (b) below:

(a) (7p) Compute the eigenspace E_3. In other words, determine all vectors $\mathbf{x}$ such that $\mathbf{Ax} = 3\mathbf{x}$.

(b) (8p) Compute the eigenspace E_6. In other words, determine all vectors $\mathbf{x}$ such that $\mathbf{Ax} = 6\mathbf{x}$.

Solution to (a): We seek to solve $(3\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{0}$. In other words

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that x_3 is free. Set $x_3 = t$ to obtain $x_1 = x_3 = t$ and $x_2 = x_3 = t$. In other words, $\mathbf{x} \in E_3$ if and only if

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$$

for some real number t.

Solution to (b): We seek to solve $(6\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{0}$. In other words

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that both x_2 and x_3 are free. Set $x_2 = s$ and $x_3 = t$. Then

$$x_1 = -x_2 - x_3 = -s - t.$$

In other words, $\mathbf{x} \in E_6$ if and only if

$$\mathbf{x} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$$

for some real numbers s and t.

Question 4: (20p) In this question, you are given four examples of a vector space V with some subset W identified. In each case, specify whether W is a *linear subspace* of V or not. Please motivate each answer (both the affirmative ones, and the negative ones). Five points per question.

(a) $V = \mathbb{R}^2$ and $W = \{\mathbf{x} = [x_1,\, x_2] \in \mathbb{R}^2 :\ x_2 \geq 0\}$.

NOT a subspace since W is not closed under scaling. To demonstrate this, consider

$$\mathbf{v} = \big[0,\, 1\big].$$

We see that $\mathbf{v} \in W$. But $(-1)\mathbf{v} = \big[0,\, -1\big] \notin W$.

(b) $V = \mathbb{R}^3$, $\mathbf{A}$ is a fixed 4×3 matrix, and $W = \{\mathbf{x} \in \mathbb{R}^3 :\ \mathbf{Ax} = 0\}$.

YES, W is a subspace. We will prove that it is closed under both addition and scaling. Then the subspace theorem asserts that W is a subspace.

Addition: Suppose that $\mathbf{x}, \mathbf{y} \in W$. Set $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then

$$\mathbf{Az} = \mathbf{A}\big(\mathbf{x} + \mathbf{y}\big) = \mathbf{Ax} + \mathbf{Ay} = \{\text{Use that } \mathbf{x}, \mathbf{y} \in W\} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

We see that $\mathbf{z} \in W$.

Scalar multiplication: Suppose that $\mathbf{x} \in W$ and $c \in \mathbb{R}$. Set $\mathbf{z} = c\mathbf{x}$. Then

$$\mathbf{Az} = \mathbf{A}\big(c\mathbf{x}\big) = c\mathbf{Ax} = \{\text{Use that } \mathbf{x} \in W\} = c\mathbf{0} = \mathbf{0}.$$

We see that $\mathbf{z} \in W$.

(c) V is the set of continuous functions on $\mathbb{R}$, that is, $V = C(\mathbb{R})$. $W = \{f \in V :\ f(0) = 1\}$.

NOT a subspace. For instance, the zero vector is not in W, so it cannot possibly be a subspace.

(d) V is the set of continuous functions on $\mathbb{R}$, that is $V = C(\mathbb{R})$. $W = \{f \in V :\ f(1) + f(2) = 0\}$.

YES, W is a subspace. We will prove that it is closed under both addition and scaling. Then the subspace theorem asserts that W is a subspace.

Addition: Suppose that $f, g \in W$. Set $h = f + g$. Then

$$\begin{aligned} h(1) + h(2) &= (f(1) + g(1)) + (f(2) + g(2)) \\ &= (f(1) + f(2)) + (g(1) + g(2)) = \{\text{Use that } f, g \in W\} = 0 + 0 = 0. \end{aligned}$$

We see that $h \in W$.

Scalar multiplication: Suppose that $f \in W$ and $c \in \mathbb{R}$. Set $h = cf$. Then

$$h(1) + h(2) = cf(1) + cf(2) = c(f(1) + f(2)) = \{\text{Use that } f \in W\} = c0 = 0.$$

We see that $h \in W$.

Question 5: (15p) Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices of size $n \times n$ that are "similar".

(a) (5p) State the definition of what it means for $\mathbf{A}$ and $\mathbf{B}$ to be "similar".

$\mathbf{A}$ and $\mathbf{B}$ are *similar* if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}.$$

(b) (5p) Prove that $\det(\mathbf{A}) = \det(\mathbf{B})$.

Assume that $\mathbf{A}$ and $\mathbf{B}$ are similar, so that $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ for some invertible $\mathbf{P}$. Then

$$\det(\mathbf{A}) = \det(\mathbf{P}\mathbf{B}\mathbf{P}^{-1}) = \det(\mathbf{P})\det(\mathbf{B})\det(\mathbf{P}^{-1}) = \det(\mathbf{P})\det(\mathbf{B})\frac{1}{\det(\mathbf{P})} = \det(\mathbf{B}).$$

(c) (5p) Prove that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

Assume that $\mathbf{A}$ and $\mathbf{B}$ are similar, so that $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ for some invertible $\mathbf{P}$.

Suppose that λ is an eigenvalue of $\mathbf{A}$ so that

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

for some non-zero vector $\mathbf{v}$. Insert $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ to get

$$\mathbf{P}\mathbf{B}\mathbf{P}^{-1}\mathbf{v} = \lambda\mathbf{v}.$$

Left multiply by $\mathbf{P}^{-1}$ to get

$$\mathbf{B}\mathbf{P}^{-1}\mathbf{v} = \lambda\mathbf{P}^{-1}\mathbf{v}.$$

Since $\mathbf{P}$ is invertible, $\mathbf{P}^{-1}\mathbf{v}$ is non-zero, and consequently an eigenvector of $\mathbf{B}$ with eigenvalue λ. So λ is also an eigenvalue of $\mathbf{B}$.

To prove that every eigenvalue of $\mathbf{B}$ is also an eigenvalue of $\mathbf{A}$, simply repeat the argument using that $\mathbf{B} = \mathbf{Q}\mathbf{A}\mathbf{Q}^{-1}$ for $\mathbf{Q} = \mathbf{P}^{-1}$.

Alternative solution: Assume that $\mathbf{A}$ and $\mathbf{B}$ are similar, so that $\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}$ for some invertible $\mathbf{P}$. We will use an argument similar to the one in (b) to prove that $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomials.

$$\begin{aligned} p_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \det(\lambda\mathbf{P}\mathbf{P}^{-1} - \mathbf{P}\mathbf{B}\mathbf{P}^{-1}) &= \det(\mathbf{P})\det(\lambda\mathbf{I} - \mathbf{B})\det(\mathbf{P}^{-1}) \\ &= \det(\mathbf{P})\det(\lambda\mathbf{I} - \mathbf{B})\frac{1}{\det(\mathbf{P})} = \det(\lambda\mathbf{I} - \mathbf{B}) = p_{\mathbf{B}}(\lambda). \end{aligned}$$

Recalling that the eigenvalues are the roots of the characteristic polynomial, it follows immediately that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

Question 6: (5p) Let $\mathbf{A}$ be a matrix of size $n \times n$ such that $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$. Prove that if n is odd, then $\det(\mathbf{A}) = 0$. Is the same statement necessarily true if n is even (please motivate)?

Solution: Suppose that $\mathbf{A}$ is an $n \times n$ matrix such that $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$. We observe that

$$\det\big(\mathbf{A}\big) = \det\big(-\mathbf{A}^{\mathrm{T}}\big) = (-1)^n \det\big(\mathbf{A}^{\mathrm{T}}\big) = (-1)^n \det\big(\mathbf{A}\big). \tag{1}$$

When n is odd, (1) implies that $\det\big(\mathbf{A}\big) = -\det\big(\mathbf{A}\big)$, which shows that $\det\big(\mathbf{A}\big) = 0$.

When n is even, there is no useful information in (1). Indeed, the statement is not true in this case, as illustrated by the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We see that $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$, while $\det\big(\mathbf{A}\big) = 1$.