Section exam 2 for M341: Linear Algebra and Matrix Theory
Thursday, March 28, 2024. 75 minutes exam time. Closed books. No notes.
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NAME: ___________________________________________________________________

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**Question 1:** (25p) In this question, we as usual let $X^T$ denote the transpose of a matrix $X$. No motivation is required for these problems.

(a) (5p) Consider the matrix $A = \begin{bmatrix} -1 & 0 & -1 \\ 4 & 2 & 7 \\ 2 & 3 & 7 \end{bmatrix}$ and the vector $B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. In answering this question, you may use that $A$ is invertible, and that $A^{-1} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix}$.

Specify the solution to the linear system $AX = B$:

$$X = A^{-1}B = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ -10 \end{bmatrix}$$

(b) (5p) With $A$ and $B$ as in (a), specify the solution to the linear system $A^T Y = B$:

$$Y = (A^T)^{-1}B = (A^{-1})^T B = \begin{bmatrix} 7 & 14 & -8 \\ 3 & 5 & -3 \\ -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ -4 \end{bmatrix}$$

(c) (5p) Evaluate the following determinant: $\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix} = 1 \cdot 5 \cdot 8 \cdot 10 = 400$

(d) (5p) Let $A$ be a $3 \times 3$ matrix such that $\det(A) = 3$. Complete the equation: $\det(2A) = 2^3 \cdot 3 = 24$

(e) (5p) In this problem, $A$ and $B$ are two square matrices of the same dimensions. Circle the statements that are necessarily true.

(i) $\det(A + B) = \det(A) + \det(B)$. FALSE

(ii) $\det(AB) = \det(A) \det(B)$. TRUE

(iii) Every matrix has at least one real eigenvalue. FALSE

(iv) If $X$ and $Y$ are two eigenvectors of $A$, then $X + Y$ is also an eigenvector of $A$. FALSE

(v) Suppose that $W$ is a subspace of a vector space $V$, and that $\{v_j\}_{j=1}^n$ is a collection of vectors in $W$. If $x$ is a linear combination of the vectors $\{v_j\}_{j=1}^n$, then $x \in W$. TRUE
Question 2: (20p) Compute all eigenvalues and eigenvectors of the matrix

\[ A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \].

Please motivate your answers briefly.

Solution: First we evaluate the characteristic polynomial of \( A \):

\[ p_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -2 \\ -1 & \lambda - 4 \end{bmatrix} = (\lambda - 1)(\lambda - 4) + 2 = \lambda^2 - 5\lambda + 6 \]

The eigenvalues are the solutions to \( p_A(\lambda) = 0 \), so

\[ \lambda = \frac{5}{2} \pm \sqrt{\frac{25}{4} - 24} = \frac{5}{2} \pm \frac{1}{2} \].

We see that the eigenvalues are 2 and 3.

Find eigenvectors for \( \lambda = 2 \): We seek the solutions to \( (2I - A)X = 0 \).

\[
\begin{bmatrix}
1 & 2 & 0 \\
-1 & -2 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

We see that \( x_2 \) is free. Setting, e.g., \( x_2 = 1 \), we find the eigenvector

\[ x = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \].

Find eigenvectors for \( \lambda = 3 \): We seek the solutions to \( (3I - A)X = 0 \).

\[
\begin{bmatrix}
2 & 2 & 0 \\
-1 & -1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 0 \\
-1 & -1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

We see that \( x_2 \) is free. Setting, e.g., \( x_2 = 1 \), we find the eigenvector

\[ x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \].
Question 3: (15p) The matrix

\[
A = \begin{bmatrix}
5 & -1 & -1 \\
-1 & 5 & -1 \\
-1 & -1 & 5
\end{bmatrix}
\]

has the eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 6 \). Show your work when answering (a) and (b) below:

(a) (7p) Compute the eigenspace \( E_3 \). In other words, determine all vectors \( x \) such that \( Ax = 3x \).

(b) (8p) Compute the eigenspace \( E_6 \). In other words, determine all vectors \( x \) such that \( Ax = 6x \).

Solution to (a): We seek to solve \((3I - A)x = 0\). In other words

\[
\begin{bmatrix}
-2 & 1 & 1 & 0 \\
1 & -2 & 1 & 0 \\
1 & 1 & -2 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & -2 & 0 \\
1 & -2 & 1 & 0 \\
-2 & 1 & 1 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & -2 & 0 \\
0 & 0 & -3 & 0 \\
0 & 0 & -3 & 0
\end{bmatrix}
\]

We see that \( x_3 \) is free. Set \( x_3 = t \) to obtain \( x_1 = x_3 = t \) and \( x_2 = x_3 = t \). In other words, \( x \in E_3 \) if and only if

\[
x = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} t
\]

for some real number \( t \).

Solution to (b): We seek to solve \((6I - A)x = 0\). In other words

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We see that both \( x_2 \) and \( x_3 \) are free. Set \( x_2 = s \) and \( x_3 = t \). Then

\[
x_1 = -x_2 - x_3 = -s - t.
\]

In other words, \( x \in E_6 \) if and only if

\[
x = \begin{bmatrix}
-s - t \\
s \\
t
\end{bmatrix} = \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix} s + \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix} t
\]

for some real numbers \( s \) and \( t \).
Question 4: (20p) In this question, you are given four examples of a vector space $V$ with some subset $W$ identified. In each case, specify whether $W$ is a linear subspace of $V$ or not. Please motivate each answer (both the affirmative ones, and the negative ones). Five points per question.

(a) $V = \mathbb{R}^2$ and $W = \{ x = [x_1, x_2] \in \mathbb{R}^2 : x_2 \geq 0 \}$.

NOT a subspace since $W$ is not closed under scaling. To demonstrate this, consider $v = [0, 1]$.

We see that $v \in W$. But $-1v = [0, -1] \notin W$.

(b) $V = \mathbb{R}^3$, $A$ is a fixed $4 \times 3$ matrix, and $W = \{ x \in \mathbb{R}^3 : Ax = 0 \}$.

YES, $W$ is a subspace. We will prove that it is closed under both addition and scaling. Then the subspace theorem asserts that $W$ is a subspace.

Addition: Suppose that $x, y \in W$. Set $z = x + y$. Then

$$Az = A(x + y) = Ax + Ay = \{ \text{Use that } x, y \in W \} = 0 + 0 = 0.$$ 

We see that $z \in W$.

Scalar multiplication: Suppose that $x \in W$ and $c \in \mathbb{R}$. Set $z = cx$. Then

$$Az = A(cx) = cAx = \{ \text{Use that } x \in W \} = c0 = 0.$$ 

We see that $z \in W$.

(c) $V$ is the set of continuous functions on $\mathbb{R}$, that is, $V = C(\mathbb{R})$. $W = \{ f \in V : f(0) = 1 \}$.

NOT a subspace. For instance, the zero vector is not in $W$, so it cannot possibly be a subspace.

(d) $V$ is the set of continuous functions on $\mathbb{R}$, that is $V = C(\mathbb{R})$. $W = \{ f \in V : f(1) + f(2) = 0 \}$.

YES, $W$ is a subspace. We will prove that it is closed under both addition and scaling. Then the subspace theorem asserts that $W$ is a subspace.

Addition: Suppose that $f, g \in W$. Set $h = f + g$. Then

$$h(1) + h(2) = (f(1) + g(1)) + (f(2) + g(2))$$
$$= (f(1) + f(2)) + (g(1) + g(2)) = \{ \text{Use that } f, g \in W \} = 0 + 0 = 0.$$ 

We see that $h \in W$.

Scalar multiplication: Suppose that $f \in W$ and $c \in \mathbb{R}$. Set $h = cf$. Then

$$h(1) + h(2) = cf(1) + cf(2) = c(f(1) + f(2)) = \{ \text{Use that } f \in W \} = c0 = 0.$$ 

We see that $h \in W$. 
**Question 5:** (15p) Let $A$ and $B$ be two matrices of size $n \times n$ that are “similar”.

(a) (5p) State the definition of what it means for $A$ and $B$ to be “similar”.

$A$ and $B$ are similar if there exists an invertible matrix $P$ such that
\[ A = PBP^{-1}. \]

(b) (5p) Prove that $\det(A) = \det(B)$.

Assume that $A$ and $B$ are similar, so that $A = PBP^{-1}$ for some invertible $P$. Then
\[
\det(A) = \det(PBP^{-1}) = \det(P)\det(B)\det(P^{-1}) = \det(P)\det(B)\frac{1}{\det(P)} = \det(B).
\]

(c) (5p) Prove that $A$ and $B$ have the same eigenvalues.

Assume that $A$ and $B$ are similar, so that $A = PBP^{-1}$ for some invertible $P$.

Suppose that $\lambda$ is an eigenvalue of $A$ so that
\[ Av = \lambda v \]
for some non-zero vector $v$. Insert $A = PBP^{-1}$ to get
\[ PBP^{-1}v = \lambda v. \]

Left multiply by $P^{-1}$ to get
\[ BP^{-1}v = \lambda P^{-1}v. \]

Since $P$ is invertible, $P^{-1}v$ is non-zero, and consequently an eigenvector of $B$ with eigenvalue $\lambda$. So $\lambda$ is also an eigenvalue of $B$.

To prove that every eigenvalue of $B$ is also an eigenvalue of $A$, simply repeat the argument using that $B = QAQ^{-1}$ for $Q = P^{-1}$.

**Alternative solution:** Assume that $A$ and $B$ are similar, so that $A = PBP^{-1}$ for some invertible $P$. We will use an argument similar to the one in (b) to prove that $A$ and $B$ have the same characteristic polynomials.

\[
p_A(\lambda) = \det(\lambda I - A) = \det(\lambda PP^{-1} - PBP^{-1}) = \det(P)\det(\lambda I - B)\det(P^{-1})
\]
\[ = \det(P)\det(\lambda I - B)\frac{1}{\det(P)} = \det(\lambda I - B) = p_B(\lambda).\]

Recalling that the eigenvalues are the roots of the characteristic polynomial, it follows immediately that $A$ and $B$ have the same eigenvalues.
**Question 6:** (5p) Let $A$ be a matrix of size $n \times n$ such that $A = -A^T$. Prove that if $n$ is odd, then $\det(A) = 0$. Is the same statement necessarily true if $n$ is even (please motivate)?

**Solution:** Suppose that $A$ is an $n \times n$ matrix such that $A = -A^T$. We observe that

$$\det(A) = \det(-A^T) = (-1)^n \det(A^T) = (-1)^n \det(A).$$

When $n$ is odd, (1) implies that $\det(A) = -\det(A)$, which shows that $\det(A) = 0$.

When $n$ is even, there is no useful information in (1). Indeed, the statement is not true in this case, as illustrated by the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We see that $A = -A^T$, while $\det(A) = 1$. 