Section exam 2 for M341: Linear Algebra and Matrix Theory

Thursday, March 28, 2024. 75 minutes exam time. *Closed books. No notes.*

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	Question 1 (25 max)	Question 2 (20 max)	Question 3 (15 max)	Question 4 (20 max)	Question 5 (15 max)	Question 6 (5 max)	Total (100 max)
Score:							

Question 1: (25p) In this question, we as usual let $\mathbf{X}^{\mathrm{T}}$ denote the *transpose* of a matrix $\mathbf{X}$. No motivation is required for these problems.

(a) (5p) Consider the matrix $\mathbf{A} = \begin{bmatrix} -1 & 0 & -1 \\ 4 & 2 & 7 \\ 2 & 3 & 7 \end{bmatrix}$ and the vector $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. In answering this question, you may use that $\mathbf{A}$ is invertible, and that $\mathbf{A}^{-1} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix}$.

Specify the solution to the linear system $\mathbf{AX} = \mathbf{B}$:

$\mathbf{X} =$

(b) (5p) With $\mathbf{A}$ and $\mathbf{B}$ as in (a), specify the solution to the linear system $\mathbf{A}^{\mathrm{T}}\mathbf{Y} = \mathbf{B}$:

$\mathbf{Y} =$

(c) (5p) Evaluate the following determinant: $\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}\right) =$

(d) (5p) Let $\mathbf{A}$ be a 3×3 matrix such that $\det\big(\mathbf{A}\big) = 3$. Complete the equation: $\det\big(2\mathbf{A}\big) =$

(e) (5p) In this problem, $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of the same dimensions.
Circle the statements that are necessarily true.

(i) $\det(\mathbf{A} + \mathbf{B}) = \det(\mathbf{A}) + \det(\mathbf{B})$.

(ii) $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

(iii) Every matrix has at least one real eigenvalue.

(iv) If $\mathbf{X}$ and $\mathbf{Y}$ are two eigenvectors of $\mathbf{A}$, then $\mathbf{X} + \mathbf{Y}$ is also an eigenvector of $\mathbf{A}$.

(v) Suppose that W is a subspace of a vector space V, and that $\{\mathbf{v}_j\}_{j=1}^n$ is a collection of vectors in W. If $\mathbf{x}$ is a linear combination of the vectors $\{\mathbf{v}_j\}_{j=1}^n$, then $\mathbf{x} \in W$.

Question 2: (20p) Compute all eigenvalues and eigenvectors of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}.$$

Please motivate your answers briefly.

Question 3: (15p) The matrix

$$\mathbf{A} = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 6$. Show your work when answering (a) and (b) below:

(a) (7p) Compute the eigenspace E_3. In other words, determine all vectors $\mathbf{x}$ such that $\mathbf{Ax} = 3\mathbf{x}$.

(b) (8p) Compute the eigenspace E_6. In other words, determine all vectors $\mathbf{x}$ such that $\mathbf{Ax} = 6\mathbf{x}$.

Question 4: (20p) In this question, you are given four examples of a vector space V with some subset W identified. In each case, specify whether W is a *linear subspace* of V or not. Please motivate each answer (both the affirmative ones, and the negative ones). Five points per question.

(a) $V = \mathbb{R}^2$ and $W = \{\mathbf{x} = [x_1, x_2] \in \mathbb{R}^2 : x_2 \geq 0\}$.

(b) $V = \mathbb{R}^3$, $\mathbf{A}$ is a fixed 4×3 matrix, and $W = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{A}\mathbf{x} = 0\}$.

(c) V is the set of continuous functions on $\mathbb{R}$, that is, $V = C(\mathbb{R})$. $W = \{f \in V : f(0) = 1\}$.

(d) V is the set of continuous functions on $\mathbb{R}$, that is $V = C(\mathbb{R})$. $W = \{f \in V : f(1) + f(2) = 0\}$.

Question 5: (15p) Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices of size $n \times n$ that are "similar".

(a) (5p) State the definition of what it means for $\mathbf{A}$ and $\mathbf{B}$ to be "similar".

(b) (5p) Prove that $\det(\mathbf{A}) = \det(\mathbf{B})$.

(c) (5p) Prove that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

Question 6: (5p) Let A be a matrix of size $n \times n$ such that $\mathsf{A} = -\mathsf{A}^{\mathrm{T}}$. Prove that if n is odd, then $\det(\mathsf{A}) = 0$. Is the same statement necessarily true if n is even (please motivate)?

Hint for Question 6: Notice how few points it is worth!