Section exam 1 for M341: Linear Algebra and Matrix Theory
Thursday, February 22, 2024. 75 minutes exam time. *Closed books. No notes.*
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SOLUTIONS

Question 1: (20p) In this question, we as usual let $\mathbf{X}^{\mathrm{T}}$ denote the *transpose* of a matrix $\mathbf{X}$.

(a) (6p) Consider the matrix $\mathbf{G} = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$. Specify the quantities indicated. Write "N/A" in case a requested quantity does not exist.

$$\mathbf{G}^2 = \begin{bmatrix} 1 & 0 \\ -4 & 9 \end{bmatrix} \qquad \mathbf{G}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$$

(b) (6p) Consider the matrix $\mathbf{H} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$. Specify the quantities indicated. Write "N/A" in case a requested quantity does not exist.

$$\mathbf{H}^2 = \text{N/A} \qquad \mathbf{H}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 0 & 1 \end{bmatrix}$$

(c) (4p) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Specify a ***symmetric*** matrix $\mathbf{B}$ and a ***skew-symmetric*** matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$.

$$\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}}) = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \mathbf{C} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

(d) (4p) Let $\mathbf{A}$ and $\mathbf{B}$ be two square invertible matrices of the same dimensions. Mark which statements are necessarily true:

	True	False
$(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$	X	
$(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$	X	
$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$	X	
$(\mathbf{AB})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$		X

Question 2: (30p) For this question, please write *only the answer*, no motivation. 5p per question.

(a) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Specify its inverse:

$$\mathbf{A}^{-1} = \frac{1}{4 \cdot 1 - 2 \cdot 3} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}.$$

(b) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 2 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

The variable x_2 is free. Set $x_2 = t$, where t is any real number. Then

$$x_4 = 2, \qquad x_3 = 3, \qquad x_2 = t, \qquad x_1 = 5 - 2t, \qquad \mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ 3 \\ 2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Circle the statements that are necessarily true.

(i) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$. TRUE

(ii) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$. TRUE

(iii) If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z} = 0$, then $\|\mathbf{x}+\mathbf{y}+\mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{z}\|^2$. TRUE

(iv) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, and the angle between $\mathbf{x}$ and $\mathbf{y}$ is θ, then $\mathbf{x} \cdot \mathbf{y} = \sin(\theta)\,\|\mathbf{x}\|\,\|\mathbf{y}\|$. FALSE

(d) Circle the statements that are necessarily true.

(i) The row echelon form (REF) of a matrix is unique. FALSE

(ii) The reduced row echelon form (RREF) of a matrix is unique. TRUE

(iii) If $\mathbf{A}$ is an $m \times n$ matrix that is row equivalent to a matrix $\mathbf{B}$ that is in row echelon form, then the rank of $\mathbf{A}$ equals the number of non-zero rows of $\mathbf{B}$. TRUE

(iv) If $\mathbf{A}$ is an $m \times n$ matrix that is row equivalent to a matrix $\mathbf{B}$ that is in row echelon form, then the rank of $\mathbf{A}$ equals the number of non-zero columns of $\mathbf{B}$. FALSE

(e) Consider the vectors $\mathbf{a} = [1,\ 0,\ 2]$ and $\mathbf{b} = [1,\ 1,\ 1]$. Specify the projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{3}{5}[1,\ 0,\ 2] = [0.6,\ 0,\ 1.2]$$

(f) Let $\mathbf{x}$ and $\mathbf{y}$ denote vectors in $\mathbb{R}^n$. You know that $\mathbf{x} \neq \mathbf{0}$. Set $t = \dfrac{\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2}{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}$. Specify all possible values of t.

$$t = \frac{\|\mathbf{x}\|^2 + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \|\mathbf{y}\|^2 + \|\mathbf{x}\|^2 - \mathbf{x}\cdot\mathbf{y} - \mathbf{y}\cdot\mathbf{x} + \|\mathbf{y}\|^2}{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2} = 2$$

Question 3: (20p) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -1 \\ 4 & 2 & 7 \\ 2 & 3 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In answering this question, you may use that $\mathbf{A}$ is invertible, and that $\mathbf{A}^{-1} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix}$

(a) (5p) Specify the solution $\mathbf{X}$ to the equation $\mathbf{AX} = \mathbf{B}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \\ -10 \end{bmatrix}$$

(b) (5p) Specify the solution $\mathbf{X}$ to the equation $\mathbf{A}^{\mathrm{T}}\mathbf{X} = \mathbf{B}$:

$$\mathbf{X} = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{B} = (\mathbf{A}^{-1})^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} 7 & 14 & -8 \\ 3 & 5 & -3 \\ -2 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \\ -4 \end{bmatrix}$$

(c) (5p) Specify the solution $\mathbf{X}$ to the equation $(\mathbf{AC})\mathbf{X} = \mathbf{B}$:

$$\mathbf{X} = \mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{B} = \{\text{Use (a)}\} = \mathbf{C}^{-1} \begin{bmatrix} 9 \\ 17 \\ -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 17 \\ -10 \end{bmatrix} = \begin{bmatrix} -11 \\ 17 \\ -10 \end{bmatrix}.$$

(d) (5p) Specify the ranks of the matrices indicated:

$\text{rank}(\mathbf{A}) = 3$ $\qquad$ $\text{rank}([\mathbf{A} \mid \mathbf{B}]) = 3$ $\qquad$ $\text{rank}(\mathbf{AC}) = 3$

Hint: (a), (b), and (c) can in principle be solved using Gaussian elimination. However, this would be time-consuming. Using the provided inverse, there is a much faster way to solve each one. (Part (c) may be slightly challenging.)

Question 4: (20p) Let p be a real number and consider the linear system

$$\begin{cases} x_1 & & +x_3 & = & 1, \\ 2x_1 & +x_2 & +(p+2)x_3 & = & 2, \\ 2x_1 & -px_2 & +x_3 & = & p+1. \end{cases}$$

(a) (10p) Consider the case where $p = 3$. Specify the solution set.
(b) (5p) For which values of p, if any, does the system have infinitely many solutions? Specify the solution set.
(c) (5p) For which values of p, if any, does the system have no solutions?

Solution: First, we compute the REF of the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & p+2 & 2 \\ 2 & -p & 1 & p+1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & p & 0 \\ 0 & -p & -1 & p-1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & p & 0 \\ 0 & 0 & p^2-1 & p-1 \end{array}\right]$$

(a) Setting $p = 3$, we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 8 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1/4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$$

So

$$x_1 = 3/4, \qquad x_2 = -3/4, \qquad x_3 = 1/4.$$

(b) For there to be a chance of infinitely many solutions, there must be a free variable. The only possibility is for x_3 to be free, which requires $p^2 - 1 = 1$, which is to say that $p = 1$ or $p = -1$. Since the number "to the right of the line" is $p - 1$, we see that infinitely many solutions is possible only when $p = 1$. In this case, we get the REF

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that x_3 is a free variable. Set $x_3 = t$, so that the solution set is:

$$x_1 = 1 - t, \qquad x_2 = -t, \qquad x_3 = t, \qquad\qquad t \in \mathbb{R}.$$

(c) We saw in part (b) that the coefficient matrix is rank deficient when $p = \pm 1$, and further that there are infinitely many solutions when $p = 1$. When $p = -1$ on the other hand we get the REF

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

This is clearly inconsistent. $\boxed{\text{The system is inconsistent when } p = -1.}$

Question 5: (10p) You know that $\mathbf{A}$ is a 4×4 matrix of rank 3, and that $\mathbf{B}$ is a 4×1 column vector. Now consider the linear system

$$\mathbf{AX} = \mathbf{B}. \tag{1}$$

The vectors

$$\mathbf{Y} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

both solve (1). In other words, $\mathbf{AY} = \mathbf{B}$ and $\mathbf{AZ} = \mathbf{B}$.

Specify the full solution set to (1). Motivate your answer briefly.

Solution: First, observe that $\mathbf{A}$ has rank 3, and that we know that $\left[\mathbf{A}|\mathbf{B}\right]$ is a consistent linear system. Since $\mathbf{A}$ is of size 4×4, it follows that the RREF of $\left[\mathbf{A}|\mathbf{B}\right]$ has precisely one free variable.

Next, let us for a given real number t set

$$\mathbf{X} = \mathbf{Y} + t\left(\mathbf{Z} - \mathbf{Y}\right).$$

Then

$$\mathbf{AX} = \mathbf{AY} + t\left(\mathbf{AZ} - \mathbf{AY}\right) = \mathbf{B} + t\left(\mathbf{B} - \mathbf{B}\right) = \mathbf{B}$$

so any vector $\mathbf{X}$ of the form $\mathbf{X} = \mathbf{Y} + t\left(\mathbf{Z} - \mathbf{Y}\right)$ is a solution. Since we know (by the argument in the first paragraph) that $\left[\mathbf{A}|\mathbf{B}\right]$ has precisely one free variable, we have found all solutions.

Answer: $\mathbf{X}$ solves $\mathbf{AX} = \mathbf{B}$ iff $\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for some $t \in \mathbb{R}$.