Question 1: (20p) In this question, we as usual let $X^T$ denote the transpose of a matrix $X$.

(a) (6p) Consider the matrix $G = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$. Specify the quantities indicated.
Write “N/A” in case a requested quantity does not exist.

$$G^2 =$$

$$G^T =$$

(b) (6p) Consider the matrix $H = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$. Specify the quantities indicated.
Write “N/A” in case a requested quantity does not exist.

$$H^2 =$$

$$H^T =$$

(c) (4p) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Specify a symmetric matrix $B$ and a skew-symmetric matrix $C$ such that $A = B + C$.

$$B =$$

$$C =$$

(d) (4p) Let $A$ and $B$ be two square invertible matrices of the same dimensions. Mark which statements are necessarily true:

<table>
<thead>
<tr>
<th>Statement</th>
<th>True</th>
<th>False</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(A^{-1})^T = (A^T)^{-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(A + B)^T = A^T + B^T$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(A^T)^T = A$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(AB)^T = A^T B^T$</td>
<td></td>
<td></td>
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</tbody>
</table>
Question 2: (30p) For this question, please write only the answer, no motivation. 5p per question.

(a) Consider the matrix \( A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \). Specify its inverse: \( A^{-1} = \)

(b) Specify the solution set to the linear system

\[
\begin{bmatrix}
1 & 2 & -2 & 0 & -1 \\
0 & 0 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(c) Circle the statements that are necessarily true.

(i) If \( x, y \in \mathbb{R}^n \), then \( |x \cdot y| \leq \|x\| \|y\| \).

(ii) If \( x, y \in \mathbb{R}^n \), then \( \|x + y\| \leq \|x\| + \|y\| \).

(iii) If \( x, y, z \in \mathbb{R}^n \), and \( x \cdot y = x \cdot z = y \cdot z = 0 \), then \( \|x + y + z\|^2 = \|x\|^2 + \|y\|^2 + \|z\|^2 \).

(iv) If \( x, y \in \mathbb{R}^2 \), and the angle between \( x \) and \( y \) is \( \theta \), then \( x \cdot y = \sin(\theta) \|x\| \|y\| \).

(d) Circle the statements that are necessarily true.

(i) The row echelon form (REF) of a matrix is unique.

(ii) The reduced row echelon form (RREF) of a matrix is unique.

(iii) If \( A \) is an \( m \times n \) matrix that is row equivalent to a matrix \( B \) that is in row echelon form, then the rank of \( A \) equals the number of non-zero rows of \( B \).

(iv) If \( A \) is an \( m \times n \) matrix that is row equivalent to a matrix \( B \) that is in row echelon form, then the rank of \( A \) equals the number of non-zero columns of \( B \).

(e) Consider the vectors \( a = [1, 0, 2] \) and \( b = [1, 1, 1] \). Specify the projection of \( b \) onto \( a \):

\[
\text{proj}_a(b) =
\]

(f) Let \( x \) and \( y \) denote vectors in \( \mathbb{R}^n \). You know that \( x \neq 0 \). Set \( t = \frac{\|x + y\|^2 + \|x - y\|^2}{\|x\|^2 + \|y\|^2} \).

Specify all possible values of \( t \).
Question 3: (20p) Consider the matrices
\[ A = \begin{bmatrix} -1 & 0 & -1 \\ 4 & 2 & 7 \\ 2 & 3 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

In answering this question, you may use that \( A \) is invertible, and that \( A^{-1} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix} \).

(a) (5p) Specify the solution \( X \) to the equation \( AX = B \):

\[ X = \]

(b) (5p) Specify the solution \( X \) to the equation \( A^T X = B \):

\[ X = \]

(c) (5p) Specify the solution \( X \) to the equation \( (AC)X = B \):

\[ X = \]

(d) (5p) Specify the ranks of the matrices indicated:

\[ \text{rank}(A) = \quad \text{rank}([A \mid B]) = \quad \text{rank}(AC) = \]

Hint: (a), (b), and (c) can in principle be solved using Gaussian elimination. However, this would be time-consuming. Using the provided inverse, there is a much faster way to solve each one. (Part (c) may be slightly challenging.)
Question 4: (20p) Let \( p \) be a real number and consider the linear system

\[
\begin{align*}
    x_1 + x_3 &= 1, \\
    2x_1 + x_2 + (p + 2)x_3 &= 2, \\
    2x_1 - px_2 + x_3 &= p + 1.
\end{align*}
\]

(a) (10p) Consider the case where \( p = 3 \). Specify the solution set.

(b) (5p) For which values of \( p \), if any, does the system have infinitely many solutions? Specify the solution set.

(c) (5p) For which values of \( p \), if any, does the system have no solutions?
**Question 5:** (10p) You know that $A$ is a $4 \times 4$ matrix of rank 3, and that $B$ is a $4 \times 1$ column vector. Now consider the linear system

$$(1) \quad AX = B.$$ 

You know that the vectors

$$Y = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

both solve (1). In other words, $AY = B$ and $AZ = B$.

Specify the full solution set to (1). Motivate your answer briefly.