Section exam 1 for M341: Linear Algebra and Matrix Theory

Thursday, February 22, 2024. 75 minutes exam time. *Closed books. No notes.*

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	Question 1 (20 max)	Question 2 (30 max)	Question 3 (20 max)	Question 4 (20 max)	Question 5 (10 max)	Total (100 max)
Score:						

Question 1: (20p) In this question, we as usual let $\mathbf{X}^{\mathrm{T}}$ denote the *transpose* of a matrix $\mathbf{X}$.

(a) (6p) Consider the matrix $\mathbf{G} = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$. Specify the quantities indicated. Write "N/A" in case a requested quantity does not exist.

$\mathbf{G}^2 =$ $\qquad\qquad$ $\mathbf{G}^{\mathrm{T}} =$

(b) (6p) Consider the matrix $\mathbf{H} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$. Specify the quantities indicated. Write "N/A" in case a requested quantity does not exist.

$\mathbf{H}^2 =$ $\qquad\qquad$ $\mathbf{H}^{\mathrm{T}} =$

(c) (4p) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

Specify a ***symmetric*** matrix $\mathbf{B}$ and a ***skew-symmetric*** matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$.

$\mathbf{B} =$ $\qquad\qquad$ $\mathbf{C} =$

(d) (4p) Let $\mathbf{A}$ and $\mathbf{B}$ be two square invertible matrices of the same dimensions. Mark which statements are necessarily true:

	True	False
$(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{-1}$		
$(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$		
$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$		
$(\mathbf{AB})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}$		

Question 2: (30p) For this question, please write *only the answer*, no motivation. 5p per question.

(a) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. Specify its inverse: $\mathbf{A}^{-1} =$

(b) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 2 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(c) Circle the statements that are necessarily true.

(i) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$.

(ii) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

(iii) If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, and $\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z} = 0$, then $\|\mathbf{x} + \mathbf{y} + \mathbf{z}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + \|\mathbf{z}\|^2$.

(iv) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, and the angle between $\mathbf{x}$ and $\mathbf{y}$ is θ, then $\mathbf{x} \cdot \mathbf{y} = \sin(\theta) \, \|\mathbf{x}\| \, \|\mathbf{y}\|$.

(d) Circle the statements that are necessarily true.

(i) The row echelon form (REF) of a matrix is unique.

(ii) The reduced row echelon form (RREF) of a matrix is unique.

(iii) If $\mathbf{A}$ is an $m \times n$ matrix that is row equivalent to a matrix $\mathbf{B}$ that is in row echelon form, then the rank of $\mathbf{A}$ equals the number of non-zero rows of $\mathbf{B}$.

(iv) If $\mathbf{A}$ is an $m \times n$ matrix that is row equivalent to a matrix $\mathbf{B}$ that is in row echelon form, then the rank of $\mathbf{A}$ equals the number of non-zero columns of $\mathbf{B}$.

(e) Consider the vectors $\mathbf{a} = \begin{bmatrix} 1, & 0, & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1, & 1, & 1 \end{bmatrix}$. Specify the projection of $\mathbf{b}$ onto $\mathbf{a}$:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) =$$

(f) Let $\mathbf{x}$ and $\mathbf{y}$ denote vectors in $\mathbb{R}^n$. You know that $\mathbf{x} \neq \mathbf{0}$. Set $t = \dfrac{\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2}{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}$.

Specify all possible values of t.

Question 3: (20p) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -1 \\ 4 & 2 & 7 \\ 2 & 3 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In answering this question, you may use that $\mathbf{A}$ is invertible, and that $\mathbf{A}^{-1} = \begin{bmatrix} 7 & 3 & -2 \\ 14 & 5 & -3 \\ -8 & -3 & 2 \end{bmatrix}$

(a) (5p) Specify the solution $\mathbf{X}$ to the equation $\mathbf{AX} = \mathbf{B}$:

$$\mathbf{X} =$$

(b) (5p) Specify the solution $\mathbf{X}$ to the equation $\mathbf{A}^{\mathrm{T}}\mathbf{X} = \mathbf{B}$:

$$\mathbf{X} =$$

(c) (5p) Specify the solution $\mathbf{X}$ to the equation $(\mathbf{AC})\mathbf{X} = \mathbf{B}$:

$$\mathbf{X} =$$

(d) (5p) Specify the ranks of the matrices indicated:

$\operatorname{rank}(\mathbf{A}) =$ $\qquad$ $\operatorname{rank}([\mathbf{A} \mid \mathbf{B}]) =$ $\qquad$ $\operatorname{rank}(\mathbf{AC}) =$

Hint: (a), (b), and (c) can in principle be solved using Gaussian elimination. However, this would be time-consuming. Using the provided inverse, there is a much faster way to solve each one. (Part (c) may be slightly challenging.)

Question 4: (20p) Let p be a real number and consider the linear system

$$\begin{cases} x_1 & & +x_3 & = & 1, \\ 2x_1 & +x_2 & +(p+2)x_3 & = & 2, \\ 2x_1 & -px_2 & +x_3 & = & p+1. \end{cases}$$

(a) (10p) Consider the case where $p = 3$. Specify the solution set.

(b) (5p) For which values of p, if any, does the system have infinitely many solutions? Specify the solution set.

(c) (5p) For which values of p, if any, does the system have no solutions?

Question 5: (10p) You know that $\mathbf{A}$ is a 4×4 matrix of rank 3, and that $\mathbf{B}$ is a 4×1 column vector. Now consider the linear system

$$\mathbf{AX} = \mathbf{B}. \tag{1}$$

You know that the vectors

$$\mathbf{Y} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Z} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

both solve (1). In other words, $\mathbf{AY} = \mathbf{B}$ and $\mathbf{AZ} = \mathbf{B}$.

Specify the full solution set to (1). Motivate your answer briefly.