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No, this is not possible. If $\mathbf{v}_3 = \mathbf{0}$, then $\mathbf{w}_3 = \mathbf{p}_3 \in V_2$. Since V_2 is spanned by $\{\mathbf{w}_1, \mathbf{w}_2\}$, this would mean that the set $\{\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3\}$ is linearly dependent. This would contradict the assumption that the vectors $\{\mathbf{w}_i\}$ form a basis.

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$\vdots$

Step t: $\mathbf{p}_t = \displaystyle\sum_{i=1}^{t-1} \frac{\mathbf{w}_t \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2}\mathbf{v}_i \qquad \mathbf{v}_t = \mathbf{w}_t - \mathbf{p}_t$.

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- $\|\mathbf{v}_t\|$ is the distance between $\mathbf{w}_t$ and $\text{span}(\mathbf{w}_i)_{i=1}^{t-1}$. *It is never zero!*

The Gram-Schmidt process with normalization of the vectors:

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