Homework set 10 — M341, Spring 2024

Due on Sunday April 21 (because of the exam on Thursday).

Hand in solutions to:
Section 4.5: 1c, 7.
Section 5.1: 25, 34.

Problem 1 (hand in): Let $N = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ be a vector of unit length (meaning that $\|N\| = 1$). Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(X) = X - N\,(N^{\mathrm{T}}X).$$

In other words, $f(X) = AX$ where $A = I - N\,N^{\mathrm{T}}$.

(a) Show that $f(f(X)) = f(X)$.
(b) Set $Y = f(X)$ and define the vector Z via $Z = X - Y$, so that $X = Y + Z$. Show that Z is parallel to N, and that Y is orthogonal to N.

Note: We will return to the map f in Chapter 6. It is knows as an "orthogonal projection" onto the plane L through the origin that has normal vector N.

Problem 2 (hand in): Let $N = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$ be a vector of unit length (meaning that $\|N\| = 1$), as in Problem 1. Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(X) = X - 2N\,(N^{\mathrm{T}}X).$$

In other words, $f(X) = AX$ where $A = I - 2N\,N^{\mathrm{T}}$.

(a) show that $f(f(X)) = X$ for all vectors X.
(b) Show that $\|f(X)\| = \|X\|$ for all vectors X. (Hint: Use that $\|AX\|^2 = (AX) \cdot (AX) = (AX)^{\mathrm{T}}AX = X^{\mathrm{T}}A^{\mathrm{T}}AX$.)
(c) (Optional) Describe in words what the geometric meaning of f is.

Optional problems: You are encouraged to work these! But do not hand in.
Section 4.5: 1a, 3, 4, 15, 18, 23, 24.
Section 5.1: 16, 21, 30, 33, 36.

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