Problem 1: Let $c$ be a real number, and consider the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$ 

(a) Let $A$ be a matrix with three rows, and consider the matrix $B = EA$. The matrix $B$ is the result of performing an elementary row operation on $A$. Which one?

(b) Specify a matrix $F$ such that $EF = I$. (In other words, $F = E^{-1}$.) Observe that such a matrix $F$ exists for every real number $c$, including $c = 0$.

Problem 2: Let $c$ be a real number such that $c \neq 0$, and consider the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}.$$ 

(a) Let $A$ be a matrix with three rows, and consider the matrix $B = EA$. The matrix $B$ is the result of performing an elementary row operation on $A$. Which one?

(b) Specify a matrix $F$ such that $EF = I$. (In other words, $F = E^{-1}$.)

(c) Set $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Prove that there cannot exist a matrix $H$ such that $GH = I$.

Problem 3: Consider the matrix

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 

(a) Let $A$ be a matrix with three rows, and consider the matrix $B = EA$. The matrix $B$ is the result of performing an elementary row operation on $A$. Which one?

(b) Specify a matrix $F$ such that $EF = I$. (In other words, $F = E^{-1}$.)

Problem 4: Consider the two linear systems

\[
\begin{align*}
\begin{cases}
  x_1 - x_2 + x_3 &= -1 \\
  x_1 - x_2 + 2x_3 &= -2 \\
  -x_1 + 2x_2 - x_3 &= 2 
\end{cases}
\quad \text{and} \quad
\begin{cases}
  y_1 - y_2 + y_3 &= -1 \\
  y_1 - y_2 + 2y_3 &= 1 \\
  -y_1 + 2y_2 - y_3 &= 1 
\end{cases}
\end{align*}
\]

Specify a matrix $A$, and column vectors $B$ and $C$ so that the two systems can be written as $AX = B$ and $AY = C$, respectively. Then form the extended matrix $[A|B \ C]$, and transform it to its reduced row echelon form. Finally, specify the solutions $X$ and $Y$. 

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