Homework set 4 — MATH 397C — Spring 2022

Due on Thursday April 21.

Problem 1: The file `hw04p01.m` provides Matlab code for sparse LU factorization of a matrix arising from the standard five-point finite difference discretization of the Laplace operator on an $(n+2) \times (n+2)$ grid.

(a) In a log-log diagram, plot execution time T_N versus problem size $N = n^2$ for the two orderings.
(b) Repeat (a), but now plot memory requirement versus N.
(c) Form a matrix corresponding to a higher order discretization of the Laplace equation. For instance, there is a nine-point stencil formed from the tensor product of the 1D stencil $\frac{1}{h^2}\left[-\frac{1}{12}, \frac{4}{3}, -\frac{5}{2}, \frac{4}{3}, -\frac{1}{12}\right]$. Repeat (a) and (b) for this matrix. (Note, there is some subtlety about the edge layer for the higher order stencil. Please ignore this. For instance, you could form the matrix for the $(n+4) \times (n+4)$ grid, and then just consider the $n^2 \times n^2$ matrix corresponding to the central square of nodes.)

Problem 2: Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be strictly diagonally dominant, let $\mathbf{f} \in \mathbb{C}^{n\times 1}$ be a vector, and consider the linear system $\mathbf{Au} = \mathbf{f}$. Prove that the Jacobi iteration will converge towards the exact solution for every starting vector.

Problem 3: In this problem, you will numerically investigate the convergence rate of the Jacobi and Gauss-Seidel iterations for a linear system arising from the discretization of our standard model problem: The Poisson equation on a square with zero Dirichlet data, discretized via a standard five-point stencil on a uniform grid.

As a starting point, you are welcome to use the script `hw04p03.m`, provided on the webpage.

(a) Try to experimentally determine the worst case rate of convergence as a function of h for the Jacobi method. For instance, your answer may be $\|\mathbf{e}_n\| \sim \beta^n$ where $\beta = 1 - 0.3h^4$. Attach a plot.
(b) Repeat (a) for Gauss-Seidel with basic column wise ordering.
(c) Repeat (a) for Gauss-Seidel with red-black ordering.
(d) Compare you answers in (a), (b), and (c) with the upper bound $\|\mathbf{e}_n\| \leq \beta^n \|\mathbf{e}_0\|$ where $\beta = \|\mathbf{R}\|$. Try a few different matrix norms and see if one performs better than the others.
(e) *[Optional:]* Consider some other discretization of the Poisson equation that is of interest to you. (For instance, finite elements, global spectral, higher order finite difference, etc.) Describe which discretization you consider, and see if you can develop an estimate of the convergence rate of Jacobi and Gauss-Seidel for your system. (Observe that the "red-black" ordering idea may or may not be applicable to the problem that you choose!)

Problem 4: *[Question 6.16 from Applied Numerical Linear Algebra by J.W. Demmel.]* A Matlab program implementing multigrid to solve the discrete model problem on a square is available at

`https://people.eecs.berkeley.edu/~demmel/ma221_Fall09/Matlab/MG_README.html`

(and also on the class canvas page as `demmelmultigrid.zip`). Start by running the demonstration (type `makemgdemo` and then `testfmgv`). Then, try running testfmg for different right-hand sides (input array b), different numbers of weighted Jacobi convergence steps before and after each recursive call to the multigrid solver (inputs jac1 and jac2), and different numbers of iterations (input iter). The software will plot the convergence rate (ratio of consecutive residuals); does this depend on the size of b? the frequencies in b? the values of jac1 and jac2? For which values of jac1 and jac2 is the solution most efficient?