Homework set 3 — MATH 397C — Spring 2022

Due on Thursday March 28. Hand in solutions to problems 1, 2, 4, and 5.

Problem 1: Recall that a function u on the interval $I = [-\pi,\, \pi]$ can often be expressed in terms of a Fourier series

$$u(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx},$$

where the Fourier coefficients $(c_n)_{n=-\infty}^{\infty}$ are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx}\, u(x)\, dx.$$

As we saw in class, we can use the fft to compute approximations to the Fourier coefficients from a set of uniform samples $(u(x_j))_{j=0}^{N-1}$, where $x_j = 2\pi j/N$ (we think of u as a periodic function on $\mathbb{R}$, so $u(x) = u(x + 2\pi)$ for all $x \in \mathbb{R}$). Given a positive integer $N > 11$, define the approximation error you incur via

$$e_N = \max_{-5 \leq j \leq 5} |c_n - c_n^{\text{approx}}|,$$

where c_n^{approx} is the approximation you get from an N-point FFT.

In this example, you will compute e_N for the following functions:

(a) $u(x) = x$ (extended to a periodic "saw" function)

(b) $u(x) = 1 - |x/\pi|$ (extended to a periodic "tent" function)

(c) $u(x) = \cos(3x)$

(d) $u(x) = \cos(30x)$

(e) $u(x) = \sin(20x)\,\big(1 - \sin(x)\,\cos^2(x)\big)$.

(f) $u(x) = e^{-\sin^2(x)}$

(g) $u(x) = e^{-100\sin^2(x)}$

Provide plots showing the error e_N as a function of N. (If you find that the rates of convergence are very different, you may want to avoid putting all the lines in the same diagram.) Briefly discuss your findings.

Note: For problems (a) – (d), you should be able to easily compute c_n exactly. For problems where you do not have an exact value of c_n, you are welcome to estimate E_N by reporting the difference to the next approximate value.

Problem 2: Let us again consider a function u with a Fourier series

$$u(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{inx},$$

where the Fourier coefficients $(c_n)_{n=-\infty}^{\infty}$ are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx}\, u(x)\, dx. \tag{1}$$

In class, we showed that if you approximate the integral in (1) by the Trapezoidal rule with a uniform grid with N points, then you end up with the DFT of the sequence $\big(u(2\pi j/N)\big)_{j=0}^{N-1}$.

The relationship between the exact Fourier coefficients $\big(c_n\big)_{n=-\infty}^{\infty}$ and the approximations computed by taking the DFT of the sequence $\big(u(2\pi j/N)\big)_{j=0}^{N-1}$ turns out to be very well understood theoretically. For this problem, I want you to do some self guided research to learn more about this topic.

Please turn in a formula that describes the approximation error precisely.

Discuss what happens in the special case where the function u is band-limited, so that $c_n = 0$ for all n such that $|n| \geq N/2$.

Problem 3: Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{C}^N$, and define the *convolution* $\mathbf{w} = \mathbf{u} * \mathbf{u} \in \mathbb{C}^N$ via

$$w_i = \sum_{j=0}^{N-1} u_{i-j}\, v_j, \qquad i \in \{0, 1, 2, \ldots, N-1\}. \tag{2}$$

The sum (3) involves entries u_k for negative values of k. To define these, simply extend u_k to an N-periodic function of k (so that $u_{-1} = u_{N-1}$, $u_{-2} = u_{N-2}$, and so on).

(a) Prove that the DFT of $\mathbf{w}$ is the entrywise multiplication of the DFTs of $\mathbf{u}$ and $\mathbf{v}$, up to a scaling constant.

(b) Write a simple Matlab script that numerically evaluates the convolution of two vectors efficiently.

(c) *[DRAFT]* Consider next the non-periodic case. In other words, we now define $\mathbf{w}$ via

$$w_i = \sum_{j=0}^{i} u_{i-j}\, v_j, \qquad i \in \{0, 1, 2, \ldots, N-1\}. \tag{3}$$

Describe how you can use the FFT to rapidly evaluate $\mathbf{w}$, and write a code that actually executes your scheme.

Problem 4: In this problem, you will be provided a list of numerical values of a certain function u. Your task is to use Fourier differentiation to estimate the derivative u' of u.

On the course webpage, you will find a file `hw3_problem4.txt` that contains the numerical values of the vector $\mathbf{u} = \big[u(x_j)\big]_{j=0}^{N-1}$, where $N = 201$, and $x_j = \frac{2\pi j}{N}$. (You can load it via `uu = load('hw3_problem4.txt')`.)

(a) Estimate numerically $u'(x_{67})$.

(b) Estimate numerically $u'(6\gamma)$ where $\gamma = 0.57721566490153286060\dots$ is the Euler constant. (In matlab, you can type `g = double(eulergamma)` to get it.)

(c) Plot that absolute values of the Fourier coefficients of **u**. Use a logarithmic scale on the y-axis. Based on this graph, roughly how many correct digits do you expect that there are in your answers to (a) and (b)?

(d) *(Voluntary extra problem)* Estimate $u'(x_{50})$ using finite difference approximations of different orders. How does the accuracy of such a method compare to Fourier differentiation?

Please describe your methodology briefly. *Note: You might be able to guess the formula for the function values. But this is not a legitimate solution technique!*

WARNING: PROBLEM 5 IS A DRAFT — THERE MIGHT BE ERRORS

Problem 5: Let $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x}| \leq 1\}$ denote the unit disc in two dimensions, and let $\Gamma = \partial\Omega$ denote its boundary. Consider the Helmholtz equation

$$\begin{cases} -\Delta u(\boldsymbol{x}) - \kappa^2 u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma, \end{cases} \tag{4}$$

where f provides the given Dirichlet data. Let (r, t) denote polar coordinates in the plane, so that

$$\boldsymbol{x} = (x_1,\, x_2) = (r\cos(t),\, r\sin(t)).$$

Then if f has the Fourier series

$$f(t) = A_0 + \sum_{n=1}^{\infty}\big(A_n\cos(nt) + B_n\sin(nt)\big),$$

the solution to (4) is

$$u(r,t) = A_0\,\frac{J_0(\kappa r)}{J_0(\kappa)} + \sum_{n=1}^{\infty}\frac{J_n(\kappa r)}{J_n(\kappa)}\big(A_n\cos(nt) + B_n\sin(nt)\big),$$

where J_n is the n'th Bessel function. (In Matlab $J_n(\kappa r) =$ `besselj(n,kappa*r)`.)

Write a code that takes as input a vector of Dirichlet data $\mathbf{f} = (f_i)_{i=0}^{N-1}$ on N equispaced points on Γ. The output should be the solution u at any given interior point.

Specifically, for the case $\kappa = 300$, and

$$f(\boldsymbol{x}) = \sqrt{1 + x_1 x_2^2}\,\sin(100\,x_1) + \cos(\sqrt{1 + x_2})$$

evaluate $u(0.25, 0.25)$.