

***M397C***

*Fast algorithms in scientific computing – theory & practice*

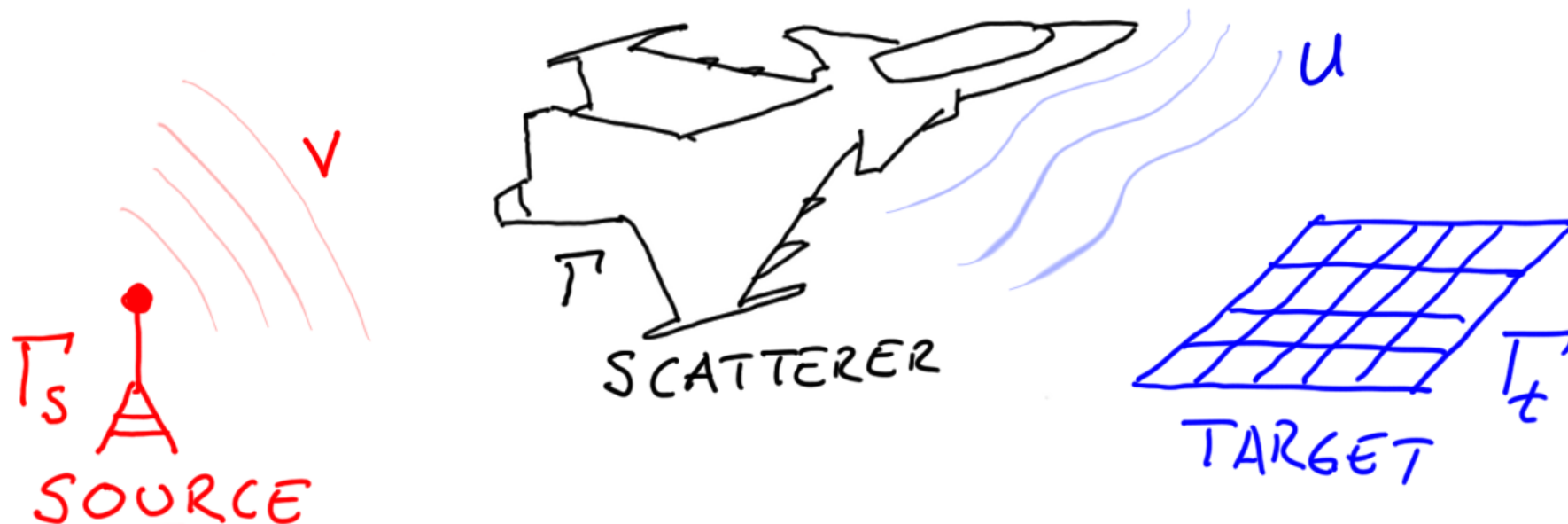
***Lecture 26: Boundary Integral Equations***

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Per-Gunnar Martinsson

The University of Texas at Austin

## Archetypical example — scattering off a finite body (here acoustic scattering):



The “sound-soft” acoustic scattering problem is to find a field  $u$  that satisfies

$$(1) \quad \begin{cases} -\Delta u(\mathbf{x}) - \kappa^2 u(\mathbf{x}) = 0 & \mathbf{x} \in \Omega^c, \\ u(\mathbf{x}) = -v(\mathbf{x}) & \mathbf{x} \in \Gamma, \\ \frac{\partial u(\mathbf{x})}{\partial |\mathbf{x}|} - i\kappa u(\mathbf{x}) = O(1/|\mathbf{x}|) & |\mathbf{x}| \rightarrow \infty, \end{cases}$$

given an “incoming field”  $v$ . One can show that (1) is “equivalent” to the equation

$$(2) \quad \frac{1}{2}\sigma(\mathbf{x}) + \int_{\Gamma} G_{\kappa}(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}') dA(\mathbf{x}') = -v(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

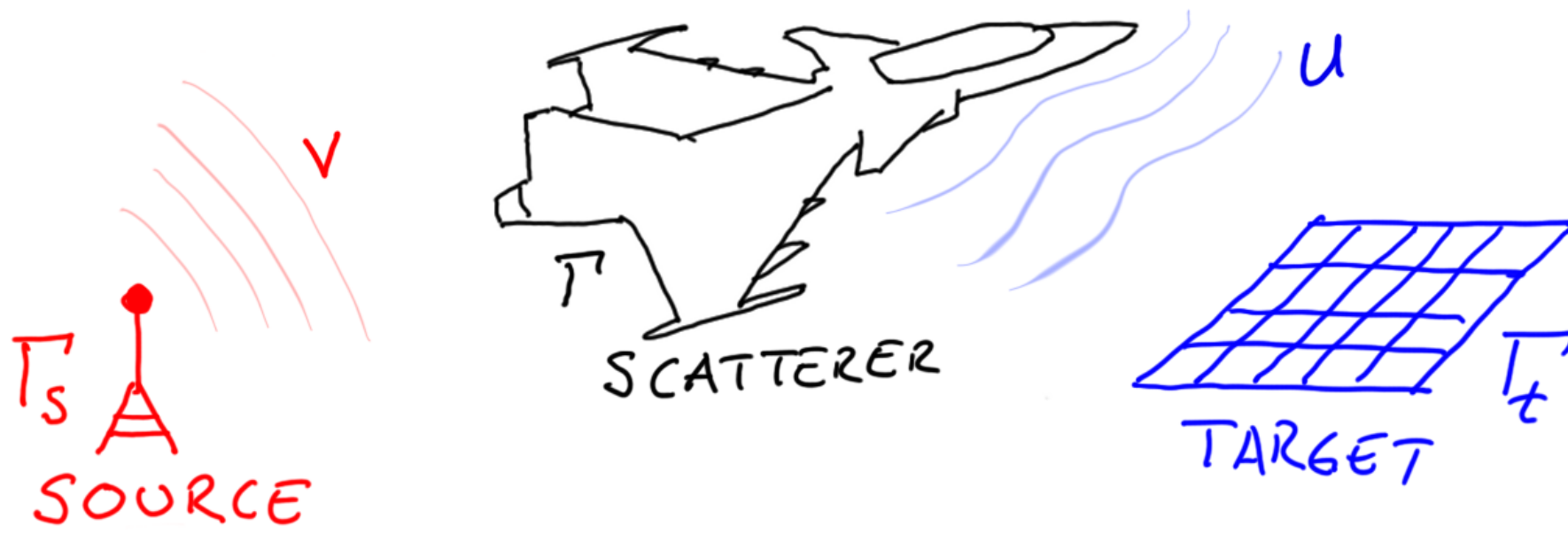
where

$$G_{\kappa}(\mathbf{x}, \mathbf{x}') = \frac{\partial \phi_{\kappa}(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}')} + i\kappa \phi_{\kappa}(\mathbf{x}, \mathbf{x}') \quad \text{and} \quad \phi_{\kappa}(\mathbf{x}, \mathbf{x}') = \frac{e^{i\kappa|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}.$$

Obvious advantages of (2): Lower dimensionality, radiation condition satisfied exactly.

Obvious disadvantage of (2): *global* operator leads to *dense* linear system.

# Archetypical example — scattering off a finite body (here acoustic scattering):



	PDE	Integral equation
Computational domain is:	infinite	finite
Computational domain is:	three dimensional	two dimensional
Radiation condition is satisfied:	approximately (hopefully)	exactly
Mathematical operator is:	unbounded	bounded (often 2 <sup>nd</sup> kind Fredholm)
Resulting linear system is:	sparse	dense

**Example — BIE for Laplace:** Consider our standard example of a BVP:

$$(3) \quad \begin{cases} -\Delta u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

Recall from the intro lecture that the exact solution operator takes the form

$$u(\mathbf{x}) = \int_{\Gamma} H(\mathbf{x}, \mathbf{y}) h(\mathbf{y}) dS(\mathbf{y}), \quad \mathbf{x} \in \Omega,$$

where  $H$  is the *Green's function* of (3). For some simple domains,  $H$  is known analytically. For instance, if  $\Omega = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \leq 1\}$ , then

$$H(r, \theta, \theta') = (2\pi)^{-1} \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - \theta')}.$$

For a general domain,  $H$  is typically not known. We look for a solution of the form

$$(4) \quad u(\mathbf{x}) = \int_{\Gamma} \phi(\mathbf{x} - \mathbf{y}) \sigma(\mathbf{y}) ds(\mathbf{y}).$$

Note that we now use a **known** kernel function — the free space fundamental solution  $\phi(\mathbf{x} - \mathbf{y}) = -\frac{1}{2\pi} \log |\mathbf{x} - \mathbf{y}|$ . The price we have to pay is that now the function  $\sigma$  is not known. But the boundary condition immediately provides an equation for  $\sigma$  — simply insert a point  $\mathbf{x} \in \Gamma$  in (4):

$$(5) \quad h(\mathbf{x}) = \int_{\Gamma} \phi(\mathbf{x} - \mathbf{y}) \sigma(\mathbf{y}) ds(\mathbf{y}), \quad \mathbf{x} \in \Gamma.$$

(Note that a function of the form (4) satisfies  $-\Delta u = 0$  for any  $\sigma$ .)

*The PDE (3) and the BIE (5) are mathematical formulations of the same problem.*

We can do better than merely achieving dimension reduction. Consider again the BVP

$$(6) \quad \begin{cases} -\Delta u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

Now look for a solution in the form of a *double layer potential*:

$$(7) \quad u(\mathbf{x}) = \int_{\Gamma} \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{2\pi|\mathbf{x} - \mathbf{y}|^2} \sigma(\mathbf{y}) ds(\mathbf{y}),$$

where  $\mathbf{n}(\mathbf{y})$  is the unit length outwards pointing normal to  $\Gamma$  at  $\mathbf{y} \in \Gamma$ .

The function  $u$  defined by (7) is discontinuous across  $\Gamma$  and for  $\mathbf{x} \in \Gamma$  we get

$$(8) \quad \underbrace{\frac{1}{2}\sigma(\mathbf{x})}_{(1/2)\text{Identity}} + \underbrace{\int_{\Gamma} \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{2\pi|\mathbf{x} - \mathbf{y}|^2} \sigma(\mathbf{y}) ds(\mathbf{y})}_{\text{Compact operator!}} = h(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

The point here is that (8) is a *second kind Fredholm equation*.

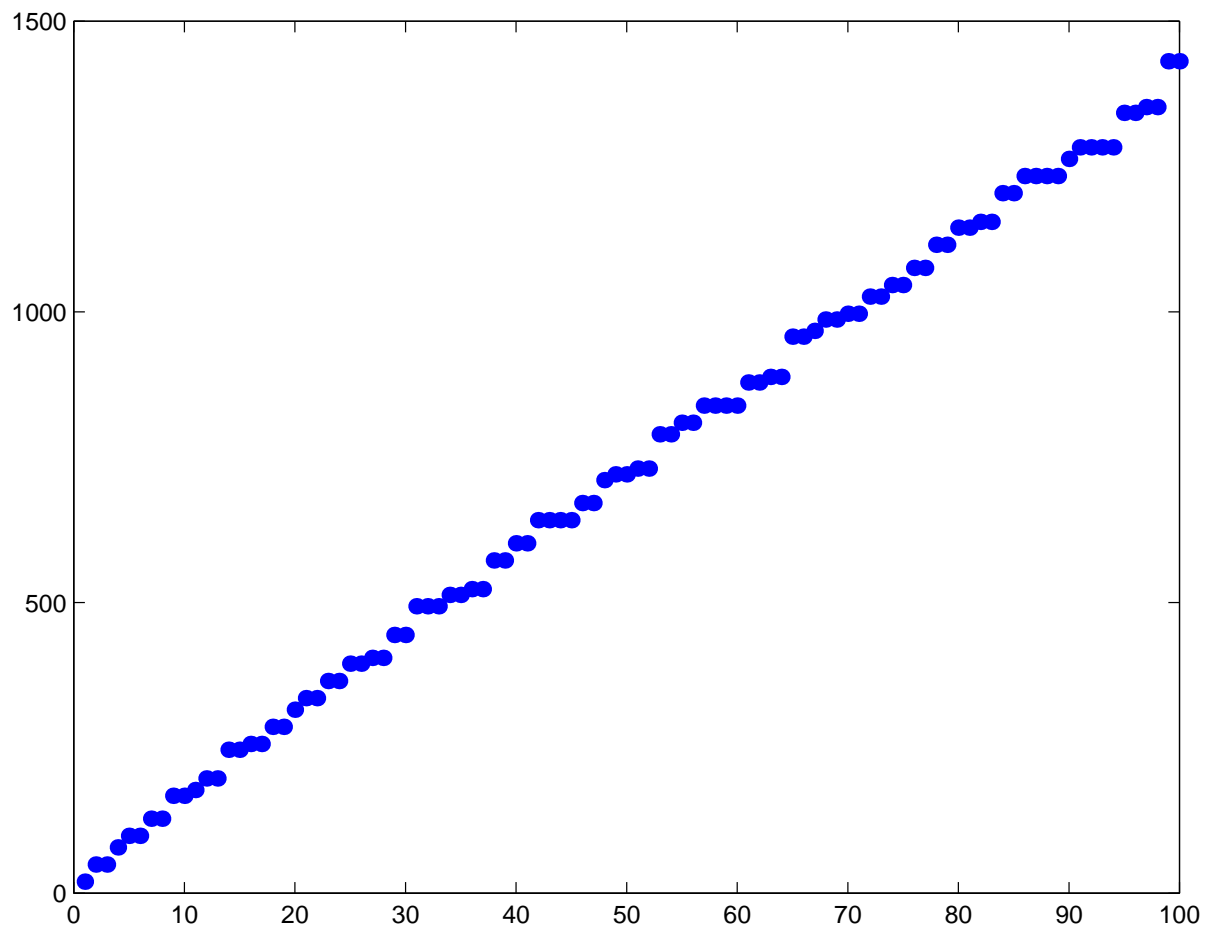
- Discretizing (8) leads to well-conditioned linear systems.
- Iterative solvers converge very fast. (Double precision in, say, 10–20 iterations is not uncommon.)

## Partial Differential Equation

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega \\ u = h & \text{on } \Gamma \end{cases}$$

$-\Delta$  is an unbounded operator.

Typical spectrum of  $-\Delta$ :



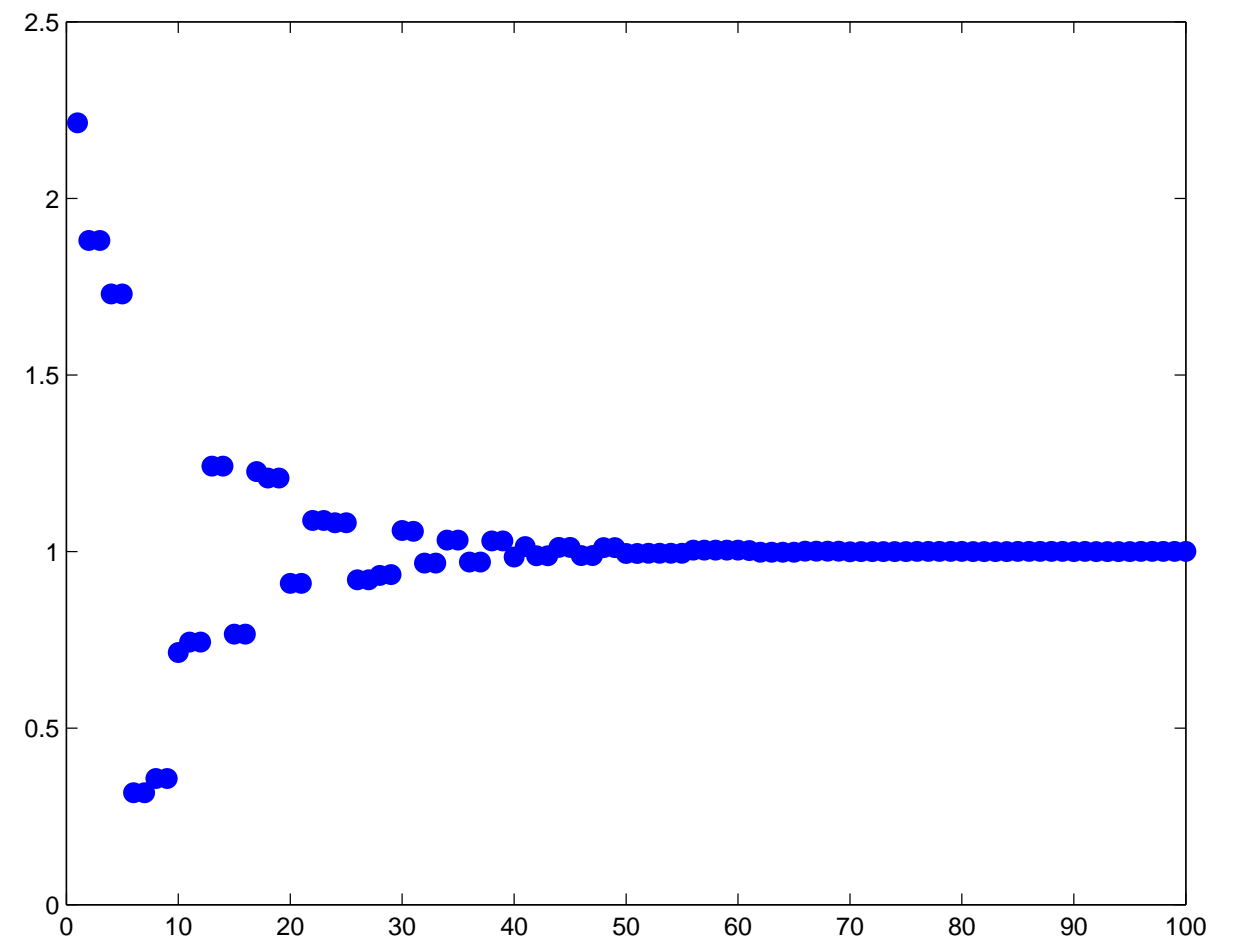
## Second kind Fredholm Eqn.

$$(I + K)q = h \quad \text{on } \Gamma$$

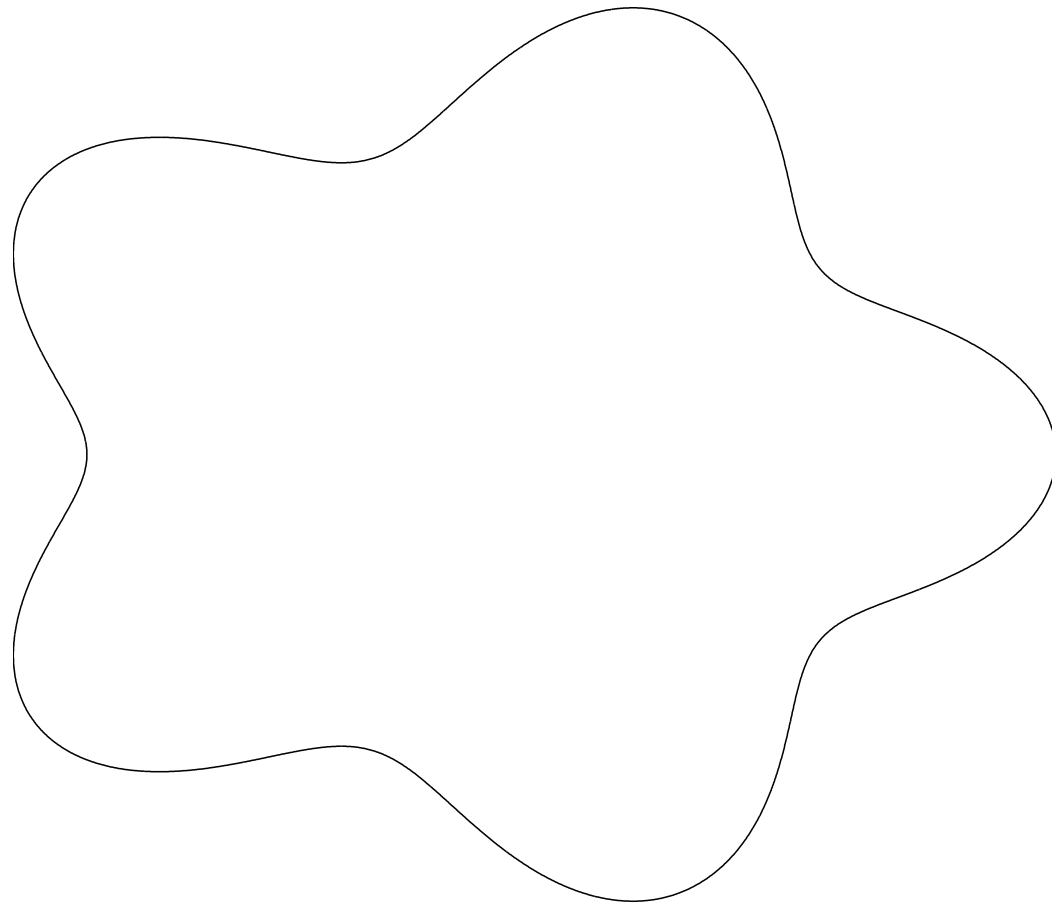
$K$  is a compact operator.

(It is “almost finite-dimensional.”)

Typical spectrum of  $I + K$ :



The condition numbers of the *discretized* operators.



$h$	Cond. nr. of discretized BIE
0.2	8.546837835256035 ( $N = 25$ )
0.1	7.053618952378199 ( $N = 50$ )
0.05	6.993154106860152 ( $N = 100$ )
0.025	6.993012937976997 ( $N = 200$ )
0.0125	6.993012936936591 ( $N = 400$ )
0.00625	6.993012936936595 ( $N = 800$ )

Rewriting the BVP as a BIE can be viewed as **analytic pre-conditioning**.

**Body loads:** Now consider a BVP with a body load

$$(9) \quad \begin{cases} -\Delta u(\mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

First construct a function  $v$  such that  $-\Delta v = g$ , disregarding boundary conditions.

This is easy:

$$(10) \quad v(\mathbf{x}) = \int_{\Omega} \phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \Omega.$$

Now look for a solution to (9) of the form  $u = v + w$ . Then  $w$  must satisfy

$$\begin{cases} -\Delta w(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ w(\mathbf{x}) = g(\mathbf{x}) - v(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

The solution  $w$  is found by solving  $\frac{1}{2}\sigma(\mathbf{x}) + \int_{\Gamma} \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{2\pi|\mathbf{x} - \mathbf{y}|^2} \sigma(\mathbf{y}) ds(\mathbf{y}) = g(\mathbf{x}) - v(\mathbf{x})$ ,

for  $\sigma$ , and then setting  $w(\mathbf{x}) = \int_{\Gamma} \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{2\pi|\mathbf{x} - \mathbf{y}|^2} \sigma(\mathbf{y}) ds(\mathbf{y})$ .

**Note:** In practice, this strategy is not entirely trivial to implement. In particular, finding high-order accurate quadratures for evaluating (10) is a bit tricky for a general domain  $\Omega$  (note that  $\phi$  is weakly singular). Then you need an FMM for evaluation, etc.



## For a given BVP, there are usually different choices of BIE:

*Example:* Recall that the BVP

$$\begin{cases} -\Delta u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

can be rewritten either using a *single layer formulation*

$$(11) \quad \int_{\Gamma} \frac{-1}{2\pi} \log |\mathbf{x} - \mathbf{y}| \sigma(\mathbf{y}) ds(\mathbf{y}) = h(\mathbf{x}), \quad \mathbf{x} \in \Gamma,$$

or a *double layer formulation*

$$(12) \quad \frac{1}{2} \sigma(\mathbf{x}) + \int_{\Gamma} \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{2\pi |\mathbf{x} - \mathbf{y}|^2} \sigma(\mathbf{y}) ds(\mathbf{y}) = h(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

The formulation (12) is a second kind Fredholm equation, which is much “nicer” than (11). (It also involves a  $C^\infty$  kernel, which is *very* nice, but this is unusual.)

This situation is very typical — there are usually several different ways of formulating a physical problem as an integral equation. Choosing the best one is important.

**BIEs for the Helmholtz equation — brief notes:** Consider the equation

$$(13) \quad \begin{cases} -\Delta u(\mathbf{x}) - \kappa^2 u(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = h(\mathbf{x}), & \mathbf{x} \in \Gamma. \end{cases}$$

Introduce boundary operators

$$[\mathcal{S}_\kappa \sigma](\mathbf{x}) = \int_\Gamma \phi_\kappa(\mathbf{x} - \mathbf{y}) \sigma(\mathbf{y}) dS(\mathbf{y}), \quad [D_\kappa \sigma](\mathbf{x}) = \int_\Gamma [\partial_{\mathbf{n}(\mathbf{y})} \phi_\kappa](\mathbf{x} - \mathbf{y}) \sigma(\mathbf{y}) dS(\mathbf{y}).$$

where  $\phi_\kappa$  is the free space fundamental solution ( $\phi_\kappa(\mathbf{x}) = \frac{i}{4} H_0^{(1)}(\kappa|\mathbf{x}|)$  in 2D and  $\phi_\kappa(\mathbf{x}) = \frac{e^{i\kappa|\mathbf{x}|}}{4\pi|\mathbf{x}|}$  in 3D). Then we can proceed as we did for Laplace, and rewrite (13) as

$$\frac{1}{2}\sigma(\mathbf{x}) + [D_\kappa \sigma](\mathbf{x}) = h(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

However, there are problems with so called *spurious resonances*. The standard fix is to look for a solution of the form

$$u(\mathbf{x}) = \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) dS(\mathbf{y}),$$

where

$$G_\kappa(\mathbf{x}, \mathbf{x}') = \frac{\partial \phi_\kappa(\mathbf{x}, \mathbf{x}')}{\partial \mathbf{n}(\mathbf{x}')} \pm i\kappa \phi_\kappa(\mathbf{x}, \mathbf{x}').$$

Then (13) can be written as a *combined field BIE*, (without spurious resonances),

$$\frac{1}{2}\sigma(\mathbf{x}) + \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{x}') \sigma(\mathbf{x}') dA(\mathbf{x}') = -v(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

## Integral equation formulations of other elliptic PDEs — brief notes

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*Time-harmonic Maxwell:* Similar to Helmholtz problem, but vector-valued, and substantially more subtle. Finding formulations without artificial resonances is a subject of current research (see, e.g., Greengard-Epstein).

*The equations of linear elasticity:* Similar to Laplace mathematically, but messier in practice. The fundamental solutions are tensor valued, for instance.

*Singular kernels:* Most BIE formulations involve singular kernels. Discretizing these to high accuracy requires some deftness. For weakly singular kernels, it is quite manageable. Strongly singular, and hyper-singular, kernels can be challenging.

*Domains with corners/edges:* At points where the boundary is not smooth, the layer potentials typically develop singularities → loss of accuracy unless special care is taken.

*Mixed boundary conditions:* Most standard formulations are derived for either pure Dirichlet, or pure Helmholtz problems. Situations where different conditions apply to different parts of the domain are more challenging.

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*Active area of research!* The ideal would be to automate this process ...

## Nyström discretization of BIEs with smooth kernel functions

At this point, we have rephrased our problem as the task of solving the BIE

$$(14) \quad \alpha q(\mathbf{x}) + \int_{\Gamma} k(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma,$$

where  $\Gamma$  is a contour in  $\mathbb{R}^2$  or a surface in  $\mathbb{R}^3$ .

Suppose first that the kernel function  $k$  is smooth, such as, e.g., the double layer kernel associated with the Laplace equation in 2D:

$$k(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{n}(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})}{2\pi |\mathbf{x} - \mathbf{y}|^2}.$$

(Note: Smooth kernels are *very* unusual! We will consider the general case shortly.)

Suppose that we are given a quadrature rule for smooth functions on  $\Gamma$ . In other words, suppose that we are given *nodes*  $\{\mathbf{x}_i\}_{i=1}^N \subset \Gamma$  and *weights*  $\{w_i\}_{i=1}^N \subset [0, \infty)$  such that

$$(15) \quad \int_{\Gamma} \varphi(\mathbf{y}) dS(\mathbf{y}) \approx \sum_{j=1}^N w_j \varphi(\mathbf{x}_j), \quad \text{for } \varphi \text{ smooth..}$$

In order to discretize (14), we first *collocate* the equation at the quadrature nodes:

$$(16) \quad \alpha q(\mathbf{x}_i) + \int_{\Gamma} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) = f(\mathbf{x}_i), \quad i = 1, 2, 3, \dots, N.$$

Then insert (15) in to (17) to approximate the integral:

$$(17) \quad \alpha q(\mathbf{x}_i) + \sum_{j=1}^N k(\mathbf{x}_i, \mathbf{x}_j) w_j q(\mathbf{x}_j) = f(\mathbf{x}_i), \quad i = 1, 2, 3, \dots, N.$$

## Examples of convenient quadrature rules (review from A. Barnett lecture):

*Trapezoidal rule:* Simple to use and extremely accurate for smooth simple contours in 2D.

*Gaussian quadrature on “panels”:* Quite simple to use. Allows for local refinement. Good for domains with corners and edges. Slight drawback in that points cluster near the edges.

*Curtis-Clenshaw quadrature:* Panel based, just like Gaussian quadrature. Theoretically, it is much less accurate (with  $p$  points per panel of “size”  $h$ , the error in C-C is of order  $h^p$ , while the error in Gaussian quadrature is  $h^{2p-1}$ ). In practice, it is indeed a bit less accurate than Gaussian quadrature, but still very good. Useful when you want nodes at the panel interfaces.

## Nyström discretization of BIEs with weakly singular kernel functions

Now consider a more realistic situation of a BIE

$$\alpha q(\mathbf{x}) + \int_{\Gamma} k(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma,$$

where  $k(\mathbf{x}, \mathbf{y})$  is smooth for  $\mathbf{x} \neq \mathbf{y}$ , and  $k(\mathbf{x}, \mathbf{y}) \sim \log |\mathbf{x} - \mathbf{y}|$  as  $\mathbf{y} \rightarrow \mathbf{x}$ .

Assume for now that  $\Gamma$  and  $f$  are smooth.

We again start with a quadrature rule  $\{\mathbf{x}_i, w_i\}_{i=1}^N$  designed for smooth functions

$$\int_{\Gamma} \varphi(\mathbf{y}) dS(\mathbf{y}) \approx \sum_{i=1}^N w_i \varphi(\mathbf{x}_i), \quad \text{for } \varphi \text{ smooth..}$$

Collocating at the quadrature nodes, we get the equation

$$\alpha q(\mathbf{x}_i) + \int_{\Gamma} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) = f(\mathbf{x}_i), \quad i = 1, 2, 3, \dots, N.$$

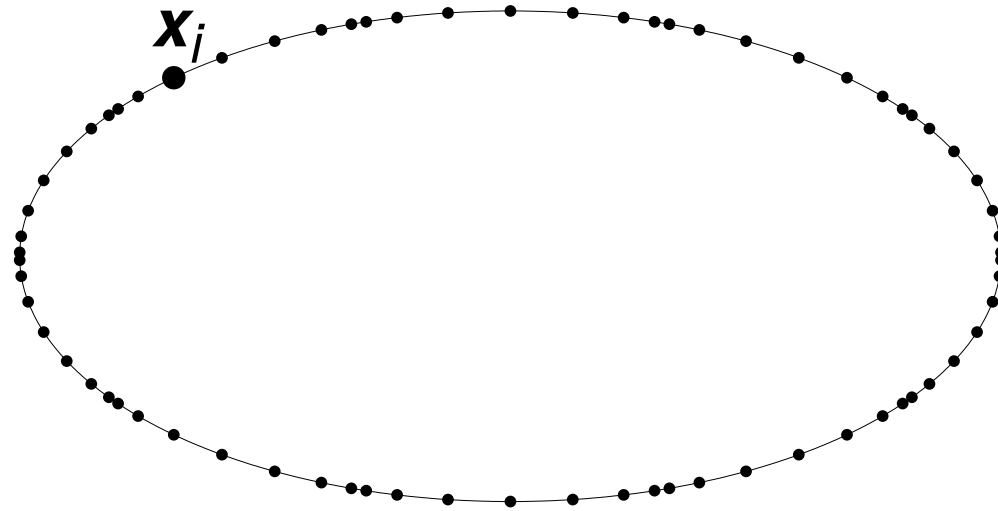
The trick is now to construct an  $N \times N$  matrix  $\mathbf{A}$  such that

$$\sum_{j=1}^N \mathbf{A}(i, j) q(\mathbf{x}_j) \approx \int_{\Gamma} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}), \quad i = 1, 2, 3, \dots, N.$$

**Key observation:** The unknown  $q$  is smooth, even though  $k$  is not.

**Recall:** We seek to build a matrix  $\mathbf{A}$  such that 
$$\sum_{j=1}^N \mathbf{A}(i,j) q(\mathbf{x}_j) \approx \int_{\Gamma} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}).$$

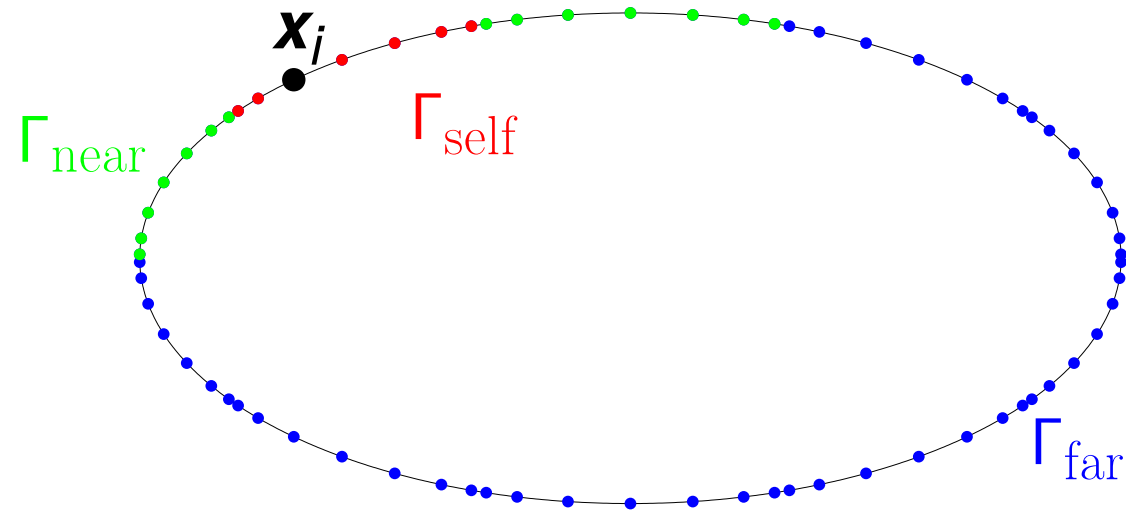
Consider the a 2D smooth contour  $\Gamma$ , and a composite Gaussian “base” quadrature:



Fix a point  $\mathbf{x}_i$ . We seek to determine the row  $\mathbf{A}(i, :)$ .

**Recall:** We seek to build a matrix  $\mathbf{A}$  such that  $\sum_{j=1}^N \mathbf{A}(i,j) q(\mathbf{x}_j) \approx \int_{\Gamma} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y})$ .

Consider the a 2D smooth contour  $\Gamma$ , and a composite Gaussian “base” quadrature:



Fix a point  $\mathbf{x}_i$ . We seek to determine the row  $\mathbf{A}(i, :)$ .

Partition  $\Gamma = \Gamma_{\text{far}} \cup \Gamma_{\text{self}} \cup \Gamma_{\text{near}}$ . Then

$$\int_{\Gamma} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) = \int_{\Gamma_{\text{far}}} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) + \int_{\Gamma_{\text{near}}} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) + \int_{\Gamma_{\text{self}}} k(\mathbf{x}_i, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}).$$

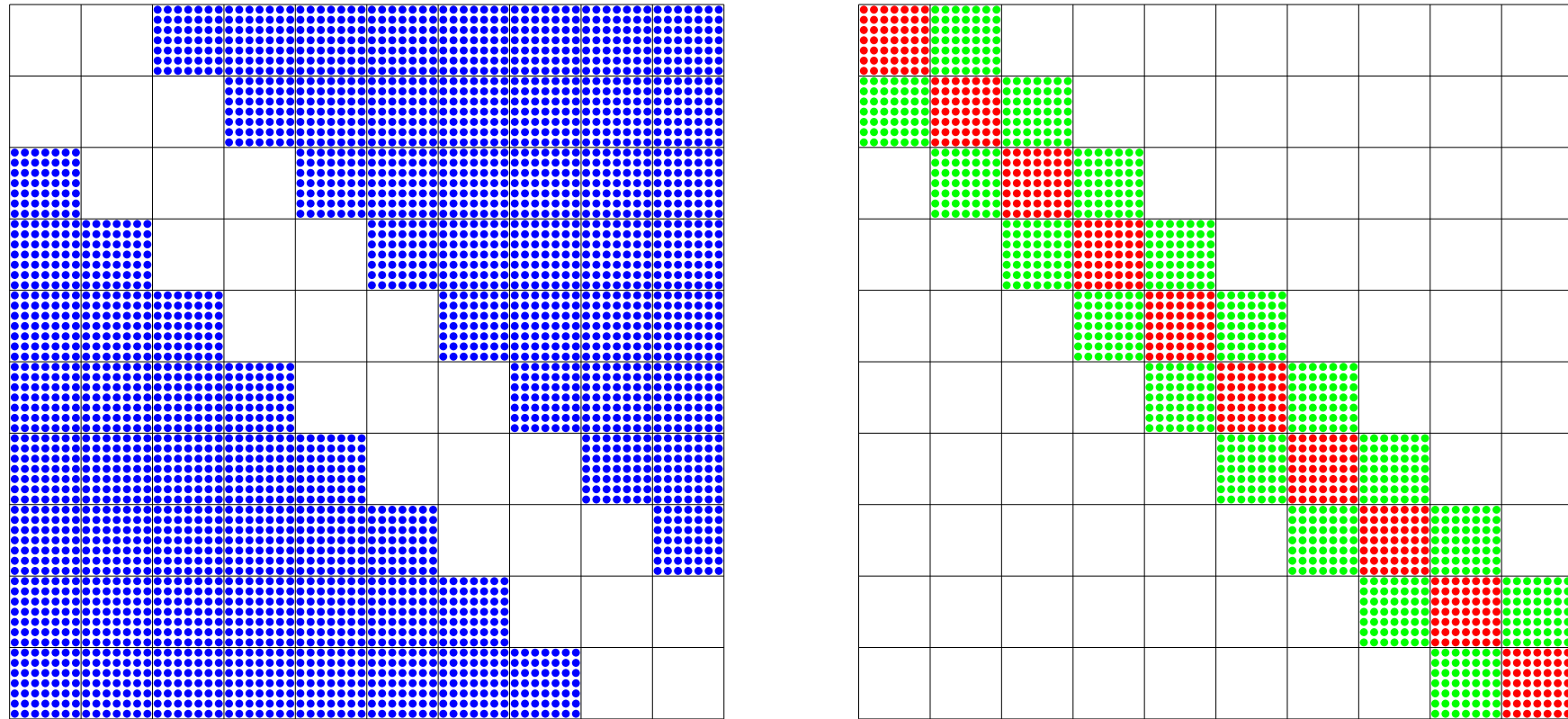
In  $\Gamma_{\text{far}}$  (which contains essentially all points), just use the plain quadrature rule.

In  $\Gamma_{\text{near}}$  and  $\Gamma_{\text{self}}$  we have to numerically build the relevant matrix blocks.



The end result is a splitting of the matrix into three parts:

$$\mathbf{A} = \mathbf{A}^{(\text{far})} + (\mathbf{A}^{(\text{near})} + \mathbf{A}^{(\text{self})})$$



All entries in  $\mathbf{A}^{(\text{far})}$  take the form  $\mathbf{A}(i, j) = k(\mathbf{x}_i, \mathbf{x}_j) w_j$ .

Analogous modifications also exist for surfaces in 3D.

*Remark:* Observe that in practice, the adjudication of whether a point is “near” is done in physical space, not in parameter space.

## Weighting of elements in building the coefficient matrix

Suppose we are given a BIE with a *smooth kernel*

$$(18) \quad \alpha q(\mathbf{x}) + \int_{\Gamma} k(\mathbf{x}, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}) = f(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

Then given a quadrature rule  $\{\mathbf{x}_i, w_i\}_{i=1}^N$  for  $\Gamma$ , the Nyström discretization of (18) is

$$(19) \quad \alpha q(\mathbf{x}_i) + \sum_{j=1}^N w_j k(\mathbf{x}_i, \mathbf{x}_j) q(\mathbf{x}_j) = f(\mathbf{x}_i), \quad i = 1, 2, 3, \dots, N.$$

To write (19) as a matrix equation, we now define vectors  $\mathbf{q}, \mathbf{f} \in \mathbb{R}^N$  via

$$\begin{aligned} \mathbf{f}(i) &= \sqrt{w_i} f(\mathbf{x}_i) && \text{given data} \\ \mathbf{q}(i) &\approx \sqrt{w_i} q(\mathbf{x}_i) && \text{sought data} \end{aligned}$$

Then define an  $N \times N$  matrix  $\mathbf{A}$  via

$$\mathbf{A}(i, j) = \sqrt{w_i} k(\mathbf{x}_i, \mathbf{x}_j) \sqrt{w_j} + \alpha \delta_{i, j}$$

and (19) can be written

$$(20) \quad \mathbf{A} \mathbf{q} = \mathbf{f}.$$

The purpose of the weighting scheme is to have  $\|\mathbf{q}\|_{\ell^2} \approx \|q\|_{L^2(\Gamma)}$  and  $\|\mathbf{f}\|_{\ell^2} \approx \|f\|_{L^2(\Gamma)}$ .

Then the singular values of  $\mathbf{A}$  approximate the singular values of  $A : L^2(\Gamma) \rightarrow L^2(\Gamma)$ .

Moreover, this avoids giving undue weight to regions where the mesh is *refined*.

## Summary:

- Many physical problems that are commonly modeled using PDEs, can advantageously be modeled using integral equations. Advantages include:
  - Reduction of dimensionality — either a PDE in 3D or a BIE on a 2D surface.
  - Bounded (often compact) operators instead of unbounded operators.
  - Second kind Fredholm formulations are often possible – excellent conditioning.
  - Radiation conditions on infinite domains enforced automatically.
- Discretization of an integral equation leads to a *dense* linear system. Linear complexity algorithms are still possible, but more memory *per degree of freedom* is usually required. (But you typically need much fewer degrees of freedom!)
- There often exist many different integral equation formulations for a given problem. Choosing the right one is tremendously helpful!
- The integral equation framework is slightly less flexible than FEM. Handling novel equations, multi-physics, non-linearities, etc, is cumbersome (at best...).
- Nyström discretization is very convenient. Often leads to a system matrix

$$\mathbf{A} = \mathbf{A}^{(\text{far})} + \mathbf{A}^{(\text{close})},$$

where  $\mathbf{A}^{(\text{close})}$  is very sparse. All elements in  $\mathbf{A}^{(\text{far})}$  take a simple form, such as,

$$\mathbf{A}^{(\text{far})}(i, j) = k(\mathbf{x}_i, \mathbf{x}_j) w_j, \quad \text{or} \quad \mathbf{A}^{(\text{far})}(i, j) = \sqrt{w_i} k(\mathbf{x}_i, \mathbf{x}_j) \sqrt{w_j}.$$