Section exam 2 for M341: Linear Algebra and Matrix Theory

9:30am – 10:45am, Nov. 17, 2022. *Closed books. No notes.*
Unique number 55415. Instructor Per-Gunnar Martinsson.

Write answers to questions 1, 2, and 3 directly on the problem sheet.
Please motivate your answers briefly, unless the question explicitly states otherwise.

Question 1: (24p) No motivations are required – full credit for the correct answer.

(a) Let V denote a vector space and let $S = \{\mathbf{v}_j\}_{j=1}^k$ denote a non-empty subset of V. State the definition of what it means for S to be *linearly independent.*

See book.

(b) Let V denote a vector space and let $B = \{\mathbf{v}_j\}_{j=1}^k$ denote a non-empty subset of V. State the definition of what it means for B to be a *basis* for V.

See book.

(c) Consider the map $T : \mathcal{P}_2 \to \mathcal{P}_3$ that maps $p \in \mathcal{P}_2$ to $q(x) = [Tp](x) = x\,p(x)$. Is T onto and/or one-to-one? Circle one option:

ONTO ONLY   [ONE-TO-ONE ONLY]   BOTH   NEITHER

(d) Consider the map $T : \mathcal{P}_3 \to \mathcal{P}_2$ that maps $p \in \mathcal{P}_3$ to $q(x) = [Tp](x) = p'(x)$ (differentiation). Is T onto and/or one-to-one? Circle one option:

[ONTO ONLY]   ONE-TO-ONE ONLY   BOTH   NEITHER

(e) Consider the map $T : \mathcal{P}_3 \to \mathcal{P}_3$ that maps $p \in \mathcal{P}_3$ to $q(x) = [Tp](x) = p'(x) + p(0)x^3$. Is T onto and/or one-to-one? Circle one option:

ONTO ONLY   ONE-TO-ONE ONLY   [BOTH]   NEITHER

(f) Let n be a positive integer, let $\mathbf{A}$ be an $n \times n$ matrix of rank $n-1$, and consider the map $T : \mathbb{R}^n \to \mathbb{R}^n$ that is given by $T\mathbf{x} = \mathbf{A}\mathbf{x}$. Is T onto and/or one-to-one? Circle one option:

ONTO ONLY   ONE-TO-ONE ONLY   BOTH   [NEITHER]

Question 2: (15p) Specify the determinants of the following matrices. No motivation required.

$$\det\left(\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\right) = \qquad \det\left(\begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 5 & 1 \end{bmatrix}\right) = \qquad \det\left(\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}\right) =$$

Answer:

(a) $1 - 6 = -5$. (Use standard cross rule.)

(b) $(-1) \times 3 \times 5 \times 2 = -30$. (Swap rows two and three and then use the rule for triangular matrices.)

(c) -6. (Do two steps of Gaussian elimination to drive the matrix to upper triangular form.)

Question 3: (20p) The matrix $\mathbf{A}$ has the three distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$ (and no other eigenvalues).

(a) Specify the eigenvalues of $\mathbf{A}^{-1}$:

(b) Specify the eigenvalues of $\mathbf{A}^{\mathrm{T}}$:

(c) Specify the eigenvalues of $\mathbf{A}^2$:

(d) Specify the eigenvalues of $\mathbf{A} + 3\mathbf{I}$:

(e) Specify the eigenvalues of $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$:

If there is not enough information provided to answer, then write "not known".

Answer:

(a) $\{1, -1/2, 1/3\}$.

(b) $\{1, -2, 3\}$.

(a) $\{1, 4, 9\}$.

(a) $\{4, 1, 6\}$.

(a) Not known.

Question 4: (15p) The matrix $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$ has an eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix}$ for some $t \in \mathbb{R}$. Specify t, and also the eigenvalue associated with $\mathbf{v}$.

Answer:

$$\mathbf{A}\mathbf{v} = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} 1-t \\ 3+t \\ -1+t \end{bmatrix} \qquad \text{and} \qquad \lambda\mathbf{v} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda t \end{bmatrix}$$

We have three equations for two unknowns. For instance observe that

$$1 - t = \lambda = 3 + t.$$

It follows that $t = -1$ so $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Then $\lambda = 1 - t = 2$.

Question 5: (20p) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 & 1 & -1 \\ 0 & 0 & -1 & 3 & 0 \\ 2 & 4 & 2 & -2 & 1 \\ 1 & 2 & -2 & 8 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{R} = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -1 & 0 & 2 \\ 2 & 2 & 1 & -1 \\ 1 & -2 & -1 & 0 \end{bmatrix}.$$

You may use that $\mathbf{A} = \mathbf{RB}$ and that $\det(\mathbf{R}) = 22$. *No motivation required for a/b/c/d.*

(a) Specify a basis for $\operatorname{ran}(\mathbf{A})$.

(b) Specify a basis for $\ker(\mathbf{A})$.

(c) Specify the dimension of $\operatorname{row}(\mathbf{A})$.

(d) Specify the dimension of $\ker(\mathbf{A}^{\mathrm{T}})$.

(e) Specify a vector $\mathbf{c} \in \mathbb{R}^4$ such that the system $\mathbf{Ax} = \mathbf{c}$ is inconsistent. *Motivation required!*

Answer:

(a) The pivots are in positions 1, 3, and 5, so the corresponding columns of $\mathbf{A}$ work:

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(b) The general solution to $\mathbf{Bx} = \mathbf{0}$ is

$$\mathbf{x} = \begin{bmatrix} -2s - 2t \\ s \\ 3t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} t,$$

so

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(c) $\dim(\operatorname{row}(\mathbf{A})) = \operatorname{rank}(\mathbf{A}) = 3$ (since there are three pivots).

(d) $\dim(\ker(\mathbf{A}^{\mathrm{T}})) = 4 - \operatorname{rank}(\mathbf{A}^{\mathrm{T}}) = 4 - \operatorname{rank}(\mathbf{A}) = 4 - 3 = 1$.

(e) Set $\mathbf{f} = [0, 0, 0, 1]^{\mathrm{T}}$. Then we know that $\mathbf{Bx} = \mathbf{f}$ is inconsistent. Then $\mathbf{RBx} = \mathbf{Rf}$ is inconsistent, which is to say that $\mathbf{Ax} = \mathbf{c}$ is inconsistent where

$$\mathbf{c} = \mathbf{Rf} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

Problem 6 (6p) *[Harder, and very few points!]* Consider the matrices

$$\mathbf{A} = \frac{1}{9}\begin{bmatrix} 17 & 2 & -2 \\ 2 & 14 & 4 \\ -2 & 4 & 14 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{V} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

In this problem, you may use that $\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$ and that $\mathbf{V}^{-1} = \mathbf{V}^{\mathrm{T}}$.

Define a sequence of vectors $\{\mathbf{x}_n\}_{n=0}^{\infty}$ in $\mathbb{R}^3$ via $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and then for $n = 1, 2, 3, \dots$, the vector $\mathbf{x}_n$ is the solution to $\mathbf{A}\mathbf{x}_n = \mathbf{x}_{n-1}$. Specify the vector $\mathbf{z} = \lim_{n\to\infty} \mathbf{x}_n$.

Solution:

Observe that

$$\mathbf{x}_n = \mathbf{A}^{-1}\mathbf{x}_{n-1},$$

so

$$\mathbf{x}_n = \mathbf{A}^{-n}\mathbf{x}_0.$$

Using that $\mathbf{A}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^{\mathrm{T}}$ we get that

$$\mathbf{x}_n = \mathbf{V}\mathbf{D}^{-n}\mathbf{V}^{\mathrm{T}}\mathbf{x}_0 = \mathbf{V}\begin{bmatrix} 1 & 0 & 0 \\ 0 & (1/2)^n & 0 \\ 0 & 0 & (1/2)^n \end{bmatrix}\mathbf{V}^{\mathrm{T}}\mathbf{x}_0.$$

Taking limits as $n \to \infty$, we get

$$\mathbf{z} = \mathbf{V}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{V}^{\mathrm{T}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{v}_1\,\mathbf{v}_1^{\mathrm{T}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}(1/3)(1-2+2) = \frac{1}{9}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

where $\mathbf{v}_1 = [1/3,\, -2/3,\, 2/3]^{\mathrm{T}}$ is the first column of $\mathbf{V}$.