Write answers to questions 1, 2, and 3 directly on the problem sheet.
Please motivate your answers briefly, unless the question explicitly states otherwise.

**Question 1:** (24p) No motivations are required – full credit for the correct answer.
(a) Let $V$ denote a vector space and let $S = \{v_j\}_{j=1}^k$ denote a non-empty subset of $V$.
State the definition of what it means for $S$ to be *linearly independent*.

(b) Let $V$ denote a vector space and let $B = \{v_j\}_{j=1}^k$ denote a non-empty subset of $V$.
State the definition of what it means for $B$ to be a *basis* for $V$.

(c) Consider the map $T : P_2 \to P_3$ that maps $p \in P_2$ to $q(x) = [Tp](x) = xp(x)$. Is $T$ onto and/or one-to-one? Circle one option:

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<th>BOTH</th>
<th>NEITHER</th>
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(d) Consider the map $T : P_3 \to P_2$ that maps $p \in P_3$ to $q(x) = [Tp](x) = p'(x)$ (differentiation). Is $T$ onto and/or one-to-one? Circle one option:

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(e) Consider the map $T : P_3 \to P_3$ that maps $p \in P_3$ to $q(x) = [Tp](x) = p'(x) + p(0)x^3$. Is $T$ onto and/or one-to-one? Circle one option:

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(f) Let $n$ be a positive integer, let $A$ be an $n \times n$ matrix of rank $n-1$, and consider the map $T : \mathbb{R}^n \to \mathbb{R}^n$ that is given by $Tx = Ax$. Is $T$ onto and/or one-to-one? Circle one option:

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**Question 2:** (15p) Specify the determinants of the following matrices. No motivation required.

\[
\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{vmatrix} = \]

\[
\begin{array}{c|c|c|c|c|c|c|c} 
 Q1 & Q2 & Q3 & Q4 & Q5 & Q6 & Total \\
(24) & (15) & (20) & (15) & (20) & (6) & (100) 
\end{array}
\]
**Question 3:** (20p) The matrix $A$ has the three distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = 3$ (and no other eigenvalues).

(a) Specify the eigenvalues of $A^{-1}$:

(b) Specify the eigenvalues of $A^T$:

(c) Specify the eigenvalues of $A^2$:

(d) Specify the eigenvalues of $A + 3I$:

(e) Specify the eigenvalues of $A + A^T$:

If there is not enough information provided to answer, then write “not known”.

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**Hand in answers to problems 4, 5, 6 on separate sheets!**

**Question 4:** (15p) The matrix $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$ has an eigenvector $v = \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix}$ for some $t \in \mathbb{R}$.

Specify $t$, and also the eigenvalue associated with $v$.

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**Question 5:** (20p) Consider the matrices $A = \begin{bmatrix} 2 & 4 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 2 & 4 & 2 & -2 \\ 1 & 2 & -2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $R = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -1 & 0 & 2 \\ 2 & 2 & 1 & -1 \\ 1 & -2 & -1 & 0 \end{bmatrix}$.

You may use that $A = RB$ and that $\det(R) = 22$. No motivation required for a/b/c/d.

(a) Specify a basis for $\text{ran}(A)$.

(b) Specify a basis for $\text{ker}(A)$.

(c) Specify the dimension of $\text{row}(A)$.

(d) Specify the dimension of $\text{ker}(A^T)$.

(e) Specify a vector $c \in \mathbb{R}^4$ such that the system $Ax = c$ is inconsistent. Motivation required!

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**Problem 6** (6p) [Harder, and very few points!] Consider the matrices $A = 1/9 \begin{bmatrix} 17 & 2 & -2 \\ 2 & 14 & 4 \\ -2 & 4 & 14 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, and $V = 1/3 \begin{bmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

In this problem, you may use that $A = VDV^{-1}$ and that $V^{-1} = V^T$.

Define a sequence of vectors $\{x_n\}_{n=0}^\infty$ in $\mathbb{R}^3$ via $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and then for $n = 1, 2, 3, \ldots$, the vector $x_n$ is the solution to $Ax_n = x_{n-1}$. Specify the vector $z = \lim_{n \to \infty} x_n$. (Full credit for the correct answer alone.)