Section exam 1 for M341: Linear Algebra and Matrix Theory

9:30am – 10:45am, Oct. 6, 2022. *Closed books. No notes.*

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Question 1: (20p) For this question, please write *only the answer*, no motivation. 5p per question.

(a) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 3 \end{bmatrix} \qquad \text{(just do back substitution)}$$

(b) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

For any real number t, there is a solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2+2t \\ -1-5t \\ t \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \end{bmatrix} t.$

(c) Let t be a real number and consider the linear system $\left[\begin{array}{ccc|c} 1 & 1 & 1 & t \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right]$.

For which value(s) of t does the linear system have a solution?

Perform Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & t \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & t \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & -t \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & t \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -t-3 \end{array}\right]$$

We see that the system is consistent iff $-t-3=0$, which is to say $\boxed{\text{for } t=-3 \text{ only}}$

(d) Circle the statements that are necessarily true.

(i) If $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$, then $\mathbf{A}$ is square and its diagonal entries are zero. T

(ii) If $\mathbf{A}$ is any matrix, and $\mathbf{B} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$, then $\mathbf{B}$ is necessarily symmetric. T

(iii) If $\mathbf{x}$ and $\mathbf{y}$ are vectors of the same dimension, then $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2$. T

(iv) If $\mathbf{A}$ is a matrix of size 4×5, then the system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has infinitely many solutions. T

(v) If $\mathbf{A}$ is a non-zero matrix of size 5×4, then there exists a vector $\mathbf{b}$ for which the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution. F

Question 2: (20p) Solve the linear system

$$\begin{array}{rrrcl} x_1 & +x_2 & -x_3 & = & 1 \\ & +2x_2 & -2x_3 & = & 4 \\ -x_1 & & +x_3 & = & 2 \end{array}$$

Please show your work.

Perform Gaussian elimination as usual:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & 2 & -2 & 4 \\ -1 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We see that

$$x_3 = 1.$$

Then

$$x_2 = 2 + x_3 = 2 + 1 = 3,$$

and

$$x_1 = 1 - x_2 + x_3 = 1 - 3 + 1 = -1.$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

Question 3: (20p) Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors of the same dimension. Prove that

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|. \tag{1}$$

You may use as a given that if $\mathbf{x}$ and $\mathbf{y}$ are both nonzero, then

$$-1 \leq \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} \leq 1. \tag{2}$$

First observe that for any vectors $\mathbf{x}$ and $\mathbf{y}$, it holds that

$$\mathbf{x}\cdot\mathbf{y} \leq \|\mathbf{x}\|\,\|\mathbf{y}\|. \tag{3}$$

When $\|\mathbf{x}\|\,\|\mathbf{y}\| \neq 0$, the relation (3) is implied by the given inequality (2), and when $\|\mathbf{x}\|\,\|\mathbf{y}\| = 0$ the relationship holds self evidently since both sides are then zero. Now

$$\|\mathbf{x}+\mathbf{y}\|^2 = (\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y}) = \mathbf{x}\cdot\mathbf{x} + \mathbf{x}\cdot\mathbf{y} + \mathbf{y}\cdot\mathbf{x} + \mathbf{y}\cdot\mathbf{y} = \|\mathbf{x}\|^2 + 2\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2. \tag{4}$$

Insert (3) into (4) to get

$$\|\mathbf{x}+\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \tag{5}$$

Take square roots of (5) to obtain (1). (Note that the quantities that are squared are necessarily non-negative, so there is no need to worry about $\pm$.)

Question 4: (20p) Consider the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

(a) (8p) Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be 3×3 matrices such that

$$\mathbf{E}_1\mathbf{A} = \begin{bmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \qquad \mathbf{E}_2\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ a_{21} & a_{22} \end{bmatrix}$$

Specify the matrices $\mathbf{E}_1$ and $\mathbf{E}_2$. (No motivation required.)

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) (8p) Specify the inverses of the matrices $\mathbf{E}_1$ and $\mathbf{E}_2$. (No motivation required.)

$$\mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{E}_2^{-1} = \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) (4p) There is a 3×3 matrix $\mathbf{F}$ such that

$$\mathbf{F}\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ 2a_{21} - 2a_{11} & 2a_{22} - 2a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Specify the matrix $\mathbf{F}$ and its inverse. (No motivation required.)

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{F}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 5: (20p) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) (4p) Compute the following quantities. No motivation required.

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(b) (4p) Let n be a positive integer and specify the quantity indicated. No motivation required.

$$\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \tag{6}$$

(c) (8p) Use induction to prove that your claim in (b) is true.

Base case: Consider $n = 1$. Then it is obviously the case that

$$\mathbf{A}^n = \mathbf{A}^1 = \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

so the claim (6) holds.

Next, suppose that (6) holds for some specific number $n = k$. We want to prove that this implies that the relation also holds for $k + 1$. We find

$$\mathbf{A}^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k \stackrel{\text{(ind)}}{=} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where in the step marked "ind", we used the induction assumption that (6) holds for $n = k$. Equation (7) is now of course (6) for $n = k + 1$.

(d) (4p) If $\mathbf{A}$ is invertible, then specify the quantities indicated. No motivation required.

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}^{-2} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{A}^{-3} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$