Section exam 1 for M341: Linear Algebra and Matrix Theory

9:30am – 10:45am, Oct. 6, 2022. *Closed books. No notes.*

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Question 1: (20p) For this question, please write *only the answer*, no motivation. 5p per question.

(a) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$

(b) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 5 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(c) Let t be a real number and consider the linear system $\left[\begin{array}{ccc|c} 1 & 1 & 1 & t \\ 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right]$.

For which value(s) of t does the linear system have a solution?

(d) Circle the statements that are necessarily true.

(i) If $\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$, then $\mathbf{A}$ is square and its diagonal entries are zero.

(ii) If $\mathbf{A}$ is any matrix, and $\mathbf{B} = \mathbf{A}^{\mathrm{T}}\mathbf{A}$, then $\mathbf{B}$ is necessarily symmetric.

(iii) If $\mathbf{x}$ and $\mathbf{y}$ are vectors of the same dimension, then $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 - 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2$.

(iv) If $\mathbf{A}$ is a matrix of size 4×5, then the system $\mathbf{A}\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

(v) If $\mathbf{A}$ is a non-zero matrix of size 5×4, then there exists a vector $\mathbf{b}$ for which the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution.

Question 2: (20p) Solve the linear system

$$\begin{array}{rrrcl} x_1 & +x_2 & -x_3 & = & 1 \\ & +2x_2 & -2x_3 & = & 4 \\ -x_1 & & +x_3 & = & 2 \end{array}$$

Please show your work.

Hint: Remember to verify your solution once you have computed it!

Question 3: (20p) Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors of the same dimension. Prove that

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

You may use as a given that if $\mathbf{x}$ and $\mathbf{y}$ are both nonzero, then

$$-1 \leq \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \leq 1.$$

Question 4: (20p) Consider the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

(a) (8p) Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be 3×3 matrices such that

$$\mathbf{E}_1\mathbf{A} = \begin{bmatrix} a_{11} + 3a_{31} & a_{12} + 3a_{32} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \qquad \mathbf{E}_2\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ a_{21} & a_{22} \end{bmatrix}$$

Specify the matrices $\mathbf{E}_1$ and $\mathbf{E}_2$. (No motivation required.)

(b) (8p) Specify the inverses of the matrices $\mathbf{E}_1$ and $\mathbf{E}_2$. (No motivation required.)

(c) (4p) There is a 3×3 matrix $\mathbf{F}$ such that

$$\mathbf{F}\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ 2a_{21} - 2a_{11} & 2a_{22} - 2a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Specify the matrix $\mathbf{F}$ and its inverse. (No motivation required.)

Hint: $\mathbf{E}_1$ and $\mathbf{E}_2$ are both standard elementary row operations, and I do not expect (a) and (b) to be hard. However, the matrix $\mathbf{F}$ is a composition of two EROs, so part (c) may by more challenging. Note how few points part (c) is worth! Do not waste time on part (c) if you are not making progress.

Question 5: (20p) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) (4p) Compute the following quantities. No motivation required.

$\mathbf{A}^2 =$ $\qquad\qquad$ $\mathbf{A}^3 =$

(b) (4p) Let n be a positive integer and specify the quantity indicated. No motivation required.

$$\mathbf{A}^n =$$

(c) (8p) Use induction to prove that your claim in (b) is true.

(d) (4p) If $\mathbf{A}$ is invertible, then specify the quantities indicated. No motivation required.

$\mathbf{A}^{-1} =$ $\qquad\qquad$ $\mathbf{A}^{-2} =$ $\qquad\qquad$ $\mathbf{A}^{-3} =$