Final exam for M341: Linear Algebra and Matrix Theory
SOLUTIONS

1:00pm – 3:00pm, Dec. 10, 2022. *Closed books. No notes.*
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Question 1: (28p) No motivations required — only the actual answer will be graded.

(a) The 5×3 matrix $\mathbf{A}$ has rank two. What is the dimension of its null space?

$$\dim(\ker(\mathbf{A})) = 3 - \text{rank}(\mathbf{A}) = 3 - 2 = 1$$

(b) Set $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & -0.5 \end{bmatrix}$. Compute the inverse of $\mathbf{A}$.

$$\mathbf{A}^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 6 \end{bmatrix}$$

(c) Determine the numbers s and t such that $\begin{bmatrix} 4 & 3 & -2 \\ 4 & 2 & -1 \\ -5 & -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & s & -1 \\ 3 & t & 4 \\ 2 & 3 & 4 \end{bmatrix}$.

$s = 0$ $\qquad$ $t = 2$

(d) Evaluate the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

$$\det(\mathbf{A}) = \det \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \{R_3 \leftarrow R_3 - R_1\} = \det \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{bmatrix} = -10$$

(e) Let $\mathbf{A}$ and $\mathbf{B}$ be square invertible matrices of the same dimension.
Circle the statements that are *always* true:

- (i) $\det(\mathbf{AB}) = \det(\mathbf{B})\det(\mathbf{A})$ TRUE
- (ii) $\det(-\mathbf{A}) = -\det(\mathbf{A})$ (not true for $n \times n$ matrices with n even)
- (iii) $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$. TRUE
- (iv) If $\mathbf{A}$ is triangular, then $\det(\mathbf{A})$ equals the product of its diagonal entries. TRUE

(f) Let $\mathbf{A}$ and $\mathbf{B}$ be square invertible matrices of the same dimension.
Circle the statements that are *always* true:

- (i) $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$. TRUE
- (ii) $(\mathbf{A} - \mathbf{I})^2 = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}$. TRUE
- (iii) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$. (only true when $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ commute)
- (iv) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$. (only true when $\mathbf{A}$ and $\mathbf{B}$ commute)

(g) Evaluate $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2$ when $\mathbf{a}$ and $\mathbf{b}$ are two vectors such that $\|\mathbf{a}\| = 1$ and $\|\mathbf{b}\| = 2$.

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = \{\text{parallelogram law}\} = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2 = 2 \cdot 1 + 2 \cdot 2^2 = 10$$

Question 2: (20p) Consider the matrices and vectors

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 6 \\ -2 \end{bmatrix}.$$

You may in solving this problem use that the RREF of $\mathbf{A}$ is $\mathbf{B}$.

(a) (12p) Specify all solutions $\mathbf{x}$ to the linear system $\mathbf{A}\mathbf{x} = \mathbf{y}$.

(b) (4p) Specify a basis for the column space (= range) of $\mathbf{A}$.

(c) (4p) Specify a basis for the null space (= kernel) of $\mathbf{A}$.

Solution:

(a) Perform Gaussian elimination on the system $\left[\mathbf{A}|\mathbf{y}\right]$ as usual

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 3 & 6 \\ -1 & 1 & 1 & -1 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that x_3 is a free variable. Set $t = x_3$. Then the solution is

$$\begin{aligned} x_1 &= 2 - t, \\ x_2 &= 1 - 2t, \\ x_3 &= t, \\ x_4 &= 1, \end{aligned}$$

where t is any real number. Alternatively,

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} t, \qquad t \in \mathbb{R}.$$

(b) The pivot columns are $\{1, 2, 4\}$. Extract the corresponding columns of $\mathbf{A}$ to get the basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

(c) Looking at the general solution in (a), we find the basis

$$\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Question 3: (20p) Compute all eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -2 \\ 6 & 5 \end{bmatrix}.$$

Please briefly motivate your computations.

Solution: We first compute the characteristic polynomial

$$p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = \det \begin{bmatrix} \lambda + 2 & 2 \\ -6 & \lambda - 5 \end{bmatrix} = (\lambda + 2)(\lambda - 5) + 12 = \lambda^2 - 3\lambda + 2.$$

Solving $p(\lambda) = 0$, we find that

$$\lambda_{1,2} = \frac{3}{2} \pm \sqrt{\frac{3^2}{2^2} - 2} = \frac{3}{2} \pm \frac{1}{2},$$

so $\lambda_1 = 1$ and $\lambda_2 = 2$. Next we look for eigenvectors.

$\boxed{\lambda_1 = 1}$ We seek all nontrivial solutions to $(\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} (1+2) & 2 & 0 \\ -6 & (1-5) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 2 & 0 \\ -6 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Set $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. (Any other solution is a scaling of $\mathbf{v}_1$.)

$\boxed{\lambda_2 = 2}$ We seek all nontrivial solutions to $(2\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} (2+2) & 2 & 0 \\ -6 & (2-5) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 2 & 0 \\ -6 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Set $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. (Any other solution is a scaling of $\mathbf{v}_2$.)

Question 4: (20p) Consider the vector space $V = \mathcal{P}_2$, consisting of all polynomials of order two or less. Please remember to briefly motivate your answers to the questions below.

(a) (5p) Is $\mathcal{B}_1 = \{1 + x - 2x^2,\ -2 + 2x + 3x^2\}$ a basis for V?

(b) (5p) Is $\mathcal{B}_2 = \{1,\ x + x^2,\ x - x^2\}$ a basis for V?

(c) (5p) Is $\mathcal{B}_3 = \{1 + x,\ 1 + x^2,\ 2 + x + x^2\}$ a basis for V?

(d) (5p) Is $\mathcal{B}_4 = \{1 + x - 2x^2,\ -2 + 3x + 3x^2,\ 3 - x + 2x^2,\ 1 + x + x^2\}$ a basis for V?

Solution:

(a) No. The dimension of V is three, so no set with only two elements can span V.

(b) The dimension of V is three, and three vectors are provided, so we need to check if they are linearly independent. Suppose that

$$0 = c_1(1) + c_2(x + x^2) + c_3(x - x^2) = 1 \times (c_1) + x \times (c_2 + c_3) + x^2 \times (c_2 - c_3).$$

So the question is whether the linear system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$$

has any non trivial solutions. One step of Gaussian elimination yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$$

so the answer is no. Since any set of three linearly independent vectors in a three dimensional space is necessarily a spanning set, we see that $\mathcal{B}_2$ is a basis.

(c) Three vectors are provided, so we need to check if they are linearly independent. Suppose that

$$0 = c_1(1 + x) + c_2(1 + x^2) + c_3(2 + x + x^2) = 1 \times (c_1 + c_2 + 2c_3) + x \times (c_1 + c_3) + x^2 \times (c_2 + c_3).$$

So the question is whether the linear system

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$$

has any non trivial solutions. Two step of Gaussian elimination yields

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0} \qquad \Rightarrow \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{0}$$

so the anser is yes, for instance $c_1 = 1$, $c_2 = 1$, $c_3 = -1$. So $\mathcal{B}_3$ is not a basis.

(d) No. The dimension of V is three, so any set with 4 vectors must be linearly dependent.

Question 5: (12p) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an orthonormal set of vectors in $\mathbb{R}^3$.

(a) (4p) Provide the definition of "orthonormal set".

(b) (4p) Let $\mathbf{a}$ be a vector such that $\mathbf{a} \cdot \mathbf{u} = 1$, $\mathbf{a} \cdot \mathbf{v} = -1$, $\mathbf{a} \cdot \mathbf{w} = 2$. What is $\|\mathbf{a}\|$?

(c) (4p) *(Harder.)* Given a vector $\mathbf{x} \in \mathbb{R}^3$, we learned in class that there is a unique vector $\mathbf{y} \in \text{span}(\mathbf{u}, \mathbf{v})$ such that $\|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x} - s\mathbf{u} - t\mathbf{v}\|$ for every $s, t \in \mathbb{R}$. Let T denote the map such that $\mathbf{y} = T(\mathbf{x})$. Is T linear? If yes, then specify a matrix $\mathbf{A}$ such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

Solution:

(a) It must be the case that

$$\mathbf{u} \cdot \mathbf{u} = 1, \quad \mathbf{u} \cdot \mathbf{v} = 0, \quad \mathbf{u} \cdot \mathbf{w} = 0, \quad \mathbf{v} \cdot \mathbf{v} = 1, \quad \mathbf{v} \cdot \mathbf{w} = 0, \quad \mathbf{w} \cdot \mathbf{w} = 1.$$

(b) Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ has three elements, and $\mathbb{R}^3$ has dimension 3, the set is not only an ON-set, but in fact an ON-*basis*. Now recall from class that if $\{\mathbf{u}_i\}_{i=1}^n$ is an ON-basis, then for any vector $\mathbf{a}$ we have

$$\|\mathbf{a}\|^2 = \sum_{i=1}^{n} \big(\mathbf{u}_i \cdot \mathbf{a}\big)^2.$$

In the present case we get

$$\|\mathbf{a}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

(c) From the information given, we know that $\mathbf{y}$ is the orthogonal projection of $\mathbf{x}$ onto the space spanned by $\{\mathbf{u}, \mathbf{v}\}$. In Chapter 6, we learned that this projection is given by the formula

$$\mathbf{y} = \big(\mathbf{u} \cdot \mathbf{x}\big)\mathbf{u} + \big(\mathbf{v} \cdot \mathbf{x}\big)\mathbf{v}.$$

Treating $\mathbf{u}$ and $\mathbf{v}$ as 3×1 matrices, we rewrite the formula as

$$\mathbf{y} = \mathbf{u}\big(\mathbf{u}^{\mathrm{T}}\mathbf{x}\big) + \mathbf{v}\big(\mathbf{v}^{\mathrm{T}}\mathbf{x}\big) = \big(\mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}}\big)\mathbf{x},$$

so the answer is yes, with

$$\mathbf{A} = \mathbf{u}\mathbf{u}^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}}.$$