Final exam for M341: Linear Algebra and Matrix Theory

1:00pm – 3:00pm, Dec. 10, 2022. *Closed books. No notes.*

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Question 1: (28p) No motivations required — only the actual answer will be graded.

(a) The 5×3 matrix $\mathbf{A}$ has rank two. What is the dimension of its null space?

$$\dim(\ker(\mathbf{A})) =$$

(b) Set $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & -0.5 \end{bmatrix}$. Compute the inverse of $\mathbf{A}$.

$$\mathbf{A}^{-1} =$$

(c) Determine the numbers s and t such that $\begin{bmatrix} 4 & 3 & -2 \\ 4 & 2 & -1 \\ -5 & -3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & s & -1 \\ 3 & t & 4 \\ 2 & 3 & 4 \end{bmatrix}$.

$s =$ $\qquad$ $t =$

(d) Evaluate the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & 1 \\ 2 & 1 & 1 \end{bmatrix}$.

$$\det(\mathbf{A}) =$$

(e) Let $\mathbf{A}$ and $\mathbf{B}$ be square invertible matrices of the same dimension.
Circle the statements that are *always* true:

- (i) $\det(\mathbf{AB}) = \det(\mathbf{B})\det(\mathbf{A})$
- (ii) $\det(-\mathbf{A}) = -\det(\mathbf{A})$
- (iii) $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$.
- (iv) If $\mathbf{A}$ is triangular, then $\det(\mathbf{A})$ equals the product of its diagonal entries.

(f) Let $\mathbf{A}$ and $\mathbf{B}$ be square invertible matrices of the same dimension.
Circle the statements that are *always* true:

- (i) $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.
- (ii) $(\mathbf{A} - \mathbf{I})^2 = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I}$.
- (iii) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.
- (iv) $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 - \mathbf{B}^2$.

(g) Evaluate $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2$ when $\mathbf{a}$ and $\mathbf{b}$ are two vectors such that $\|\mathbf{a}\| = 1$ and $\|\mathbf{b}\| = 2$.

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 =$$

Q1 (28)	Q2 (20)	Q3 (20)	Q4 (20)	Q5 (12)	Total (100)

Question 2: (20p) Consider the matrices and vectors

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 6 \\ -2 \end{bmatrix}.$$

You may in solving this problem use that the RREF of $\mathbf{A}$ is $\mathbf{B}$.

(a) (12p) Specify all solutions $\mathbf{x}$ to the linear system $\mathbf{A}\mathbf{x} = \mathbf{y}$.

(b) (4p) Specify a basis for the column space (= range) of $\mathbf{A}$.

(c) (4p) Specify a basis for the null space (= kernel) of $\mathbf{A}$.

Hint: When solving (a), you may want to use plain Gaussian elimination as taught in the class. The RREF of $\mathbf{A}$ *may be useful to help you check that you did not commit arithmetic errors, however.*

Question 3: (20p) Compute all eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & -2 \\ 6 & 5 \end{bmatrix}.$$

Please briefly motivate your computations.

Question 4: (20p) Consider the vector space $V = \mathcal{P}_2$, consisting of all polynomials of order two or less. Please remember to briefly motivate your answers to the questions below.

(a) (5p) Is $\mathcal{B}_1 = \{1 + x - 2x^2,\ -2 + 2x + 3x^2\}$ a basis for V?

(b) (5p) Is $\mathcal{B}_2 = \{1,\ x + x^2,\ x - x^2\}$ a basis for V?

(c) (5p) Is $\mathcal{B}_3 = \{1 + x,\ 1 + x^2,\ 2 + x + x^2\}$ a basis for V?

(d) (5p) Is $\mathcal{B}_4 = \{1 + x - 2x^2,\ -2 + 3x + 3x^2,\ 3 - x + 2x^2,\ 1 + x + x^2\}$ a basis for V?

Question 5: (12p) Let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be an orthonormal set of vectors in $\mathbb{R}^3$.

(a) (4p) Provide the definition of "orthonormal set".

(b) (4p) Let $\mathbf{a}$ be a vector such that $\mathbf{a} \cdot \mathbf{u} = 1$, $\mathbf{a} \cdot \mathbf{v} = -1$, $\mathbf{a} \cdot \mathbf{w} = 2$. What is $\|\mathbf{a}\|$?

(c) (4p) *(Harder.)* Given a vector $\mathbf{x} \in \mathbb{R}^3$, we learned in class that there is a unique vector $\mathbf{y} \in \operatorname{span}(\mathbf{u}, \mathbf{v})$ such that $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x} - s\mathbf{u} - t\mathbf{v}\|$ for every $s, t \in \mathbb{R}$. Let T denote the map such that $\mathbf{y} = T(\mathbf{x})$. Is T linear? If yes, then specify a matrix $\mathbf{A}$ such that $T(\mathbf{x}) = \mathbf{A}\mathbf{x}$.