

Homework set 5 — CSE 383C / CS 383C / M 383E / ME 397, Fall 2021

Hand in solutions to: 17.1, 18.1, 19.1, from the book.

**Problem 1:** Suppose that you are working on a computer that satisfies axiom (13.7) from the book for real valued  $x$  and  $y$ , so that for any  $x, y \in \mathbf{F}$ , we have

$$x \otimes y = (x * y) (1 + \epsilon)$$

for some  $\epsilon$  such that  $|\epsilon| \leq \epsilon_{\text{mach}}$ . Suppose now that you use the real valued floating point arithmetic to define complex floating point arithmetic. For instance for  $x_1, x_2, y_1, y_2 \in \mathbf{F}$ , you define complex numbers  $x = x_1 + ix_2$  and  $y = y_1 + iy_2$ , and then complex multiplication via

$$m(x, y) = ((x_1 \otimes y_1) \ominus (x_2 \otimes y_2)) \oplus i((x_1 \otimes y_2) \oplus (x_2 \otimes y_1)).$$

Prove that

$$m(x, y) = xy(1 + \delta)$$

for some  $\delta$  such that

$$|\delta| \leq C\epsilon_{\text{mach}} + O(\epsilon_{\text{mach}}^2).$$

You do not need to get the optimal value of  $C$ . (But there is of course also no need to be wasteful!)

**Problem 2 (optional – do not hand in):** Consider the same setup as in Problem 1, but now for the complex division:

$$d(x, y) = (((x_1 \otimes y_1) \oplus (x_2 \otimes y_2)) \oplus i((x_2 \otimes y_1) \ominus (x_1 \otimes y_2))) \oplus ((y_1 \otimes y_1) \oplus (y_2 \otimes y_2)).$$

What can you say about the stability of  $d$ ?

**Problem 3:** Consider the evaluation of the complex multiplication

$$(1) \quad z = (a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Observe that formula (1) is mathematically equivalent to

$$(2) \quad z = (ac - bd) + i[(a + b)(c + d) - ac - bd].$$

The point here is that (1) involves 4 multiplications, while (2) involves 3 multiplications. This is a minor saving for scalars, but does make a difference for matrices. Suppose that  $a, b, c, d \in \mathbb{R}^{m \times m}$ , and work out how many floating point additions and multiplications (count them separately) are required to evaluate the product  $z = (a + ib)(c + id)$  using formulas (1) and (2), respectively.

**Problem 4 (optional — do not hand in):** Is there a difference in numerical stability between (1) and (2)? It may be helpful to consider what happens if  $a = c = \theta$  and  $d = b = 1/\theta$  for a real number  $\theta$  that is very large.