Problem 1: Consider the vector space $X = \mathbb{C}^m$ with the $\ell^\infty$-norm $\|x\| = \max |x_i|$. What is the corresponding induced norm of a matrix? Hint: We solved this problem in class for the $\ell^1$ norm. The argument for the $\ell^\infty$ norm is very similar.

Problem 2: Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$  

Compute an estimate to the norm on $A$ induced by the norm $\ell^p$ for $p = 1.3$. Explain your methodology.

Problem 3: Let $\lambda$ be a real number such that $\lambda \in (0,1)$, and let $a$ and $b$ be two non-negative real numbers. Prove that

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda) b,$$

with equality iff $a = b$. Hint: Consider $b = 0$ first. When $b \neq 0$, change variables to $t = a/b$.

Problem 4: [Hölder’s inequality] Suppose that $p$ is a real number such that $1 < p < \infty$, and let $q$ be such that $p^{-1} + q^{-1} = 1$. Let $f, g \in \mathbb{C}^m$ and prove that

$$|f \cdot g| \leq \|f\|_p \|g\|_q.$$  

Prove that equality holds iff $\alpha |f(i)|^p = \beta |g(i)|^q$ for some $\alpha, \beta$ such that $\alpha \beta \neq 1$.

Hint: Consider first the case where $\|f\|_p = 0$ or $\|g\|_q = 0$. For the case $\|f\|_p \|g\|_q \neq 0$, use (1) with

$$a = \left(\frac{|f(i)|}{\|f\|_p}\right)^p, \quad b = \left(\frac{|g(i)|}{\|g\|_q}\right)^q, \quad \lambda = \frac{1}{p}.$$  

Problem 3: [Minkowski’s inequality] Prove that for $p \in [1,\infty]$, and for $f, g \in \mathbb{C}^m$, we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$  

Hint: Consider the cases $p = 1, \infty$ separately. For $p \in (1,\infty)$, note that

$$|f(i) + g(i)|^p \leq (|f(i)| + |g(i)|) |f(i) + g(i)|^{p-1}.$$  

Then sum both sides of (3) and apply (2) to the right hand side.