Section exam 2 for M341 (55060) Spring 2021

Released: Sunday April 18.

Due: 5pm, Thursday April 22. This is a strict deadline. Please allow yourself a margin!

Submission logistics: Submit through GradeScope. Please ensure that you know how this works well before the deadline in case difficulties arise.

Rules:

- This is an open book exam.
- The exam should be worked individually. Unlike the homeworks, you are *not* allowed to collaborate.
- You are allowed to use calculators, computers, etc, if you find them helpful. None of the questions should require extensive calculations. For the questions where motivations are required, you should at a minimum describe the steps that you took to compute the answer.
- Motivate your work unless a question specifically states that you do not have to.
- Write your answer inside the box given. This is important for GradeScope to be able to correctly scan your exam.

Question 1: (18p) Let $\mathbf{A} = \begin{bmatrix} -3 & -2 & 4 \\ 2 & 2 & -2 \\ -2 & -1 & 3 \end{bmatrix}$. Compute all eigenvalues of $\mathbf{A}$. For each eigenvalue λ, specify the dimension k of the corresponding eigenspace E_λ, and identify k linearly independent eigenvectors that span E_λ. Briefly motivate your answers.

Answer: First we determine the eigenvalues of A.

$$p_A(\lambda) = \det \begin{bmatrix} \lambda+3 & 2 & -4 \\ -2 & \lambda-2 & 2 \\ 2 & 1 & \lambda-3 \end{bmatrix} =$$

$$= (\lambda+3)(\lambda-2)(\lambda-3) + 8 + 8 - 2(\lambda+3) + 4(\lambda-3) + 8(\lambda-2) =$$

$$= (\lambda^2-9)(\lambda-2) - 18 + 10\lambda = \lambda^3 - 2\lambda^2 - 9\lambda + 18 - 18 + 10\lambda = \lambda^3 - 2\lambda^2 + \lambda$$

The roots of $\lambda(\lambda^2 - 2\lambda + 1) = 0$ are $\lambda_1 = 0$ and $\lambda_2 = 1$.

$\boxed{\lambda_1 = 0}$ We seek the sol^{ns} to $(A - 0I)x = 0$

free variable!

$$\left[\begin{array}{ccc|c} -3 & -2 & 4 & 0 \\ 2 & 2 & -2 & 0 \\ -2 & -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -3 & -2 & 4 & 0 \\ -2 & -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So E_{λ_1} has dimension 1, and $E_{\lambda_1} = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

$\boxed{\lambda_2 = 1}$ We seek the sol^{ns} to $(A - I)x = 0$

free variables!

$$\left[\begin{array}{ccc|c} -4 & -2 & 4 & 0 \\ 2 & 1 & -2 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ -2 & -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So E_{λ_2} has dimension 2, and

$$E_{\lambda_2} = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Question 2: (20p) In all questions below, $\mathbf{A}$ is a square matrix. The subquestions are *independent*, so any assumptions on $\mathbf{A}$ made in one subquestion does not apply in any other.

(a) Suppose that $\mathbf{A}$ has an eigenvector v associated with an eigenvalue λ. Is it necessarily the case that v is an eigenvector of $\mathbf{A}^3 v = \mu v$? If yes, then specify μ.

$\boxed{\text{Yes.}}$

If $Av = \lambda v$, then $A^3 v = A^2(Av) = A^2(\lambda v) = A(\lambda^2 v) = \lambda^3 v$.

$$\boxed{\mu = \lambda^3}$$

(b) Suppose that $\mathbf{A}$ is invertible, and has an eigenvector v associated with an eigenvalue λ. Is it necessarily the case that $(\mathbf{A}^{-1} + 2\mathbf{I})v = \mu v$? If yes, then specify μ.

$\boxed{\text{Yes}}$ If $Av = \lambda v$, then $\frac{1}{\lambda} v = A^{-1} v$, so

$(A^{-1} + 2I)v = A^{-1} v + 2v = \frac{1}{\lambda} v + 2v = \left(\frac{1}{\lambda} + 2\right) v.$

$$\boxed{\mu = \frac{1}{\lambda} + 2}$$

(c) Suppose that $\mathbf{A}^3 = \mathbf{A}$. What are the possible eigenvalues of $\mathbf{A}$?

If $Av = \lambda v$, and $A^3 = A$, then

$\lambda v = Av = A^3 v = \lambda^3 v$ so $(\lambda - \lambda^3) v = 0$.

Since $v \neq 0$, λ must satisfy $\lambda - \lambda^3 = 0$.

$$\boxed{\lambda \in \{-1, 0, 1\}}$$

(d) Suppose that v is an eigenvector of $\mathbf{A}$. Is it necessarily the case that v is an eigenvector of $\mathbf{A}^{\mathrm{T}}$?

$\boxed{\text{No.}}$

Say $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an evec since $Av = v$.

However, $A^T v = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so v is <u>not</u> an evec of A^T.

(e) Suppose that $\mathbf{A}$ is invertible, and that $\mathbf{A}^{-1} = \mathbf{A}^{\mathrm{T}}$. What are the possible eigenvalues of $\mathbf{A}$?

Suppose $Av = \lambda v$ and $A^{-1} = A^T$. Then

$\|Av\|^2 = (Av)^T (Av) = v^T A^T A v = \{A^T = A^{-1}\} = v^T v = \|v\|^2$ (i)

$\|Av\|^2 = \|\lambda v\|^2 = \lambda^2 \|v\|^2$ (ii)

Since $\|v\| \neq 0$, (i) & (ii) imply $\lambda^2 = 0$.

$$\boxed{\lambda \in \{-1, 1\}}$$

Question 3: (20p) Let $V = \mathcal{P}_2$ denote the vector space consisting of all polynomials of degree two or less. (In the questions below, we use the standard shorthand that "x" denotes the variable in a polynomial, so that, e.g. $2+x$ is brief notation for the polynomial p for which $p(x) = 2+x$.)

(a) Let s be a real number, and consider the set $\mathcal{A} = \{1+x+x^2,\ 1+2x+x^2,\ 1+x+sx^2\}$. For which values of s is $\mathcal{A}$ a spanning set for V?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & s \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & s-1 \end{bmatrix}$$

$\boxed{s \neq 1}$

(b) Let t be a real number, and consider the set $\mathcal{B} = \{1+tx,\ x+tx^2,\ t+x^2\}$. For which values of t is $\mathcal{B}$ a basis for V?

Set $T = \begin{bmatrix} 1 & 0 & t \\ t & 1 & 0 \\ 0 & t & 1 \end{bmatrix}$. Then $\mathcal{B}$ is a spanning set iff T has full rank.

$\det(T) = 1+t^3+0-0-0-0 = 1+t^3$

The only root of $1+t^3=0$ is $t=-1$

$\boxed{t \neq -1}$

(c) Let u be a real number, and consider the set $\mathcal{C} = \{1+2x+ux^2,\ u^2+3x+4x^2,\ u-x+u^2x^2,\ 5-2x+ux^2\}$. For which values of u is $\mathcal{C}$ a linearly independent set?

V has dimension 3.
$\mathcal{C}$ has 4 elements. $\Rightarrow$ $\boxed{\mathcal{C} \text{ is NEVER linearly indep}}$

(d) Let U be the set of all polynomials that are *derivatives* of polynomials in V. Specify a basis for U.

Let $p = a+bx+cx^2$ denote a general vector in V.

Then $p' = b+2cx$, so $U = \mathcal{P}_1$.

$\boxed{\text{The set } \{1, x\} \text{ is a basis for } U}$

Question 4: (18p) Set $\mathbf{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{A}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -4 \end{bmatrix}$, $\mathbf{A}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{A}_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$,

and $\mathbf{A} = [\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4\ \mathbf{A}_5] = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & 3 & 0 & 1 \\ -2 & 0 & -4 & -1 & 0 \end{bmatrix}$. The RREF of $\mathbf{A}$ is $\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) The "range" L of $\mathbf{A}$ is the vector space that is formed by all linear combinations of the columns of $\mathbf{A}$. In order words, $L = \text{span}\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5\}$. Specify a basis for L.

We see from the RREF that $\{1,2,4\}$ are pivot columns.

So $\{A_1, A_2, A_4\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is a basis for L.

(b) Specify a basis for the row space of $\mathbf{A}$.

Recall that row(A) = row(B) when A and B are row equivalent.

The first three rows of B are lin. indep., so they form a basis.

$\boxed{\{[1,0,2,0,-1], [0,1,-1,0,2], [0,0,0,1,1]\} \text{ is a basis.}}$

(c) Let s and t be real numbers and set $\mathbf{C} = [2 \quad -3 \quad s \quad 1 \quad t]$. For which values of s and t is it the case that $\mathbf{C}$ is in the row space of $\mathbf{A}$?

$C \in \text{row}(A) = \text{row}(B)$ if and only if $\text{rref}\begin{bmatrix} B \\ \hline C \end{bmatrix} = \begin{bmatrix} B \\ \hline 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & -3 & s & 1 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & -3 & s-4 & 1 & t+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & s-7 & 1 & t+11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & s-7 & 0 & t+9 \end{bmatrix}$$

So $C \in \text{row}(A) \iff s = 7$ and $t = -9$

Question 5: (18p) Suppose that you are given the task of solving a general fourth order polynomial equation

(1) $$x^4 + c_3\,x^3 + c_2\,x^2 + c_1\,x + c_0 = 0.$$

Prove that the task of finding the roots of (1) is equivalent to finding the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 \end{bmatrix}.$$

Solution: We compute the char. polynomial of A:

$$p_A(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ c_0 & c_1 & c_2 & \lambda + c_3 \end{bmatrix} = \{\text{Assume } \lambda \neq 0\} =$$

$$= \det\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & c_1 + \frac{c_0}{\lambda} & c_2 & \lambda + c_3 \end{bmatrix} =$$

$$= \det\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & c_2 + \frac{1}{\lambda}c_1 + \frac{1}{\lambda^2}c_0 & \lambda + c_3 \end{bmatrix} =$$

$$= \det\begin{bmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda + c_3 + \frac{1}{\lambda}c_2 + \frac{1}{\lambda^2}c_1 + \frac{1}{\lambda^3}c_0 \end{bmatrix} =$$

$$= \lambda^3\left(\lambda + c_3 + \frac{1}{\lambda}c_2 + \frac{1}{\lambda^2}c_1 + \frac{1}{\lambda^3}c_0\right) = \lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0$$

For $\lambda = 0$: $$p_A(0) = \det\begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ c_0 & c_1 & c_2 & c_3 \end{bmatrix} = (-1)^3 \det\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = (-1)^3(-1)^3 c_0 = c_0$$

So $p_A(\lambda) = \lambda^4 + c_3\lambda^3 + c_2\lambda^2 + c_1\lambda + c_0 \Rightarrow \lambda$ is eval $\Leftrightarrow \lambda$ solves (1)

Question 6: (6p) *[Only six points!]* Consider the matrix $\mathbf{A} = \begin{bmatrix} 0.52 & 0.36 & -0.20 \\ 0.36 & 0.73 & 0.15 \\ -0.20 & 0.15 & 0.25 \end{bmatrix}$.

Specify the limiting matrix $\mathbf{B} = \lim_{n\to\infty} \mathbf{A}^n$.

Hint: The matrix $\mathbf{A}$ *can be written as* $\mathbf{A} = \mathbf{U}\mathbf{U}^{\mathrm{T}} + \mathbf{V}\mathbf{V}^{\mathrm{T}}$ *where* $\mathbf{V} = \begin{bmatrix} -0.4 \\ 0.3 \\ 0.5 \end{bmatrix}$ *is an eigenvector of* $\mathbf{A}$, *and where* $\mathbf{U}$ *is another eigenvector that you would need to determine.*

Solution: We first determine the vector U from the hint:

$$UU^T = A - VV^T = \begin{bmatrix} 0.36 & 0.48 & 0 \\ 0.48 & 0.64 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} u_1^2 = 0.36 \\ u_2^2 = 0.64 \\ u_3^2 = 0 \end{matrix} \Rightarrow U = \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix}.$$

How does this help? Let us compute some powers of A:

$$A^2 = (UU^T + VV^T)(UU^T + VV^T) = UU^TUU^T + UU^TVV^T + VV^TUU^T + VV^TVV^T.$$

Observe:

$$U^TU = \|U\|^2 = 1$$
$$V^TV = \|V\|^2 = 1/2$$
$$U^TV = V^TU = 0$$

So

$$A^2 = U\underbrace{U^TU}_{=1}U^T + U\underbrace{U^TV}_{=0}V^T + V\underbrace{V^TU}_{=0}U^T + V\underbrace{V^TV}_{=1/2}V^T = UU^T + \frac{1}{2}VV^T$$

Similarly: $A^3 = (UU^T + \frac{1}{2}VV^T)(UU^T + VV^T) = UU^T + \frac{1}{4}VV^T$

By induction: $A^n = UU^T + \frac{1}{2^{n-1}}VV^T$

Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$, we find

$$\lim_{n\to\infty} A^n = UU^T = \begin{bmatrix} 0.36 & 0.48 & 0 \\ 0.48 & 0.64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$