

## Section exam 2 for M341 (55060) Spring 2021

**Released:** Sunday April 18.

**Due:** 5pm, Thursday April 22. This is a strict deadline. Please allow yourself a margin!

**Submission logistics:** Submit through GradeScope. Please ensure that you know how this works well before the deadline in case difficulties arise.

**Rules:**

- This is an open book exam.
- The exam should be worked individually. Unlike the homeworks, you are *not* allowed to collaborate.
- You are allowed to use calculators, computers, etc, if you find them helpful. None of the questions should require extensive calculations. For the questions where motivations are required, you should at a minimum describe the steps that you took to compute the answer.
- Motivate your work unless a question specifically states that you do not have to.
- Write your answer inside the box given. This is important for GradeScope to be able to correctly scan your exam.

**Question 1:** (18p) Let  $\mathbf{A} = \begin{bmatrix} -3 & -2 & 4 \\ 2 & 2 & -2 \\ -2 & -1 & 3 \end{bmatrix}$ . Compute all eigenvalues of  $\mathbf{A}$ . For each eigenvalue  $\lambda$ , specify the dimension  $k$  of the corresponding eigenspace  $E_\lambda$ , and identify  $k$  linearly independent eigenvectors that span  $E_\lambda$ . Briefly motivate your answers.

Answer:

**Question 2:** (20p) In all questions below,  $\mathbf{A}$  is a square matrix. The subquestions are *independent*, so any assumptions on  $\mathbf{A}$  made in one subquestion does not apply in any other.

(a) Suppose that  $\mathbf{A}$  has an eigenvector  $v$  associated with an eigenvalue  $\lambda$ . Is it necessarily the case that  $v$  is an eigenvector of  $\mathbf{A}^3 v = \mu v$ ? If yes, then specify  $\mu$ .

(b) Suppose that  $\mathbf{A}$  is invertible, and has an eigenvector  $v$  associated with an eigenvalue  $\lambda$ . Is it necessarily the case that  $(\mathbf{A}^{-1} + 2\mathbf{I})v = \mu v$ ? If yes, then specify  $\mu$ .

(c) Suppose that  $\mathbf{A}^3 = \mathbf{A}$ . What are the possible eigenvalues of  $\mathbf{A}$ ?

(d) Suppose that  $v$  is an eigenvector of  $\mathbf{A}$ . Is it necessarily the case that  $v$  is an eigenvector of  $\mathbf{A}^T$ ?

(e) Suppose that  $\mathbf{A}$  is invertible, and that  $\mathbf{A}^{-1} = \mathbf{A}^T$ . What are the possible eigenvalues of  $\mathbf{A}$ ?

**Question 3:** (20p) Let  $V = \mathcal{P}_2$  denote the vector space consisting of all polynomials of degree two or less. (In the questions below, we use the standard shorthand that “ $x$ ” denotes the variable in a polynomial, so that, e.g.  $2 + x$  is brief notation for the polynomial  $p$  for which  $p(x) = 2 + x$ .)

(a) Let  $s$  be a real number, and consider the set  $\mathcal{A} = \{1 + x + x^2, 1 + 2x + x^2, 1 + x + sx^2\}$ . For which values of  $s$  is  $\mathcal{A}$  a spanning set for  $V$ ?

(b) Let  $t$  be a real number, and consider the set  $\mathcal{B} = \{1 + tx, x + tx^2, t + x^2\}$ . For which values of  $t$  is  $\mathcal{B}$  a basis for  $V$ ?

(c) Let  $u$  be a real number, and consider the set  $\mathcal{C} = \{1 + 2x + ux^2, u^2 + 3x + 4x^2, u - x + u^2x^2, 5 - 2x + ux^2\}$ . For which values of  $u$  is  $\mathcal{C}$  a linearly independent set?

(d) Let  $U$  be the set of all polynomials that are *derivatives* of polynomials in  $V$ . Specify a basis for  $U$ .

**Question 4:** (18p) Set  $\mathbf{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}$ ,  $\mathbf{A}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{A}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -4 \end{bmatrix}$ ,  $\mathbf{A}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ ,  $\mathbf{A}_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ,

and  $\mathbf{A} = [\mathbf{A}_1 \ \mathbf{A}_2 \ \mathbf{A}_3 \ \mathbf{A}_4 \ \mathbf{A}_5] = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & 3 & 0 & 1 \\ -2 & 0 & -4 & -1 & 0 \end{bmatrix}$ . The RREF of  $\mathbf{A}$  is  $\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ .

(a) The “range”  $L$  of  $\mathbf{A}$  is the vector space that is formed by all linear combinations of the columns of  $\mathbf{A}$ . In other words,  $L = \text{span}\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5\}$ . Specify a basis for  $L$ .

(b) Specify a basis for the row space of  $\mathbf{A}$ .

(c) Let  $s$  and  $t$  be real numbers and set  $\mathbf{C} = \begin{bmatrix} 2 & -3 & s & 1 & t \end{bmatrix}$ . For which values of  $s$  and  $t$  is it the case that  $\mathbf{C}$  is in the row space of  $\mathbf{A}$ ?

**Question 5:** (18p) Suppose that you are given the task of solving a general fourth order polynomial equation

(1) 
$$x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0 = 0.$$

Prove that the task of finding the roots of (1) is equivalent to finding the eigenvalues of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 \end{bmatrix}.$$

Solution:

**Question 6:** (6p) *[Only six points!]* Consider the matrix  $\mathbf{A} = \begin{bmatrix} 0.52 & 0.36 & -0.20 \\ 0.36 & 0.73 & 0.15 \\ -0.20 & 0.15 & 0.25 \end{bmatrix}$ .

Specify the limiting matrix  $\mathbf{B} = \lim_{n \rightarrow \infty} \mathbf{A}^n$ .

*Hint: The matrix  $\mathbf{A}$  can be written as  $\mathbf{A} = \mathbf{U}\mathbf{U}^T + \mathbf{V}\mathbf{V}^T$  where  $\mathbf{V} = \begin{bmatrix} -0.4 \\ 0.3 \\ 0.5 \end{bmatrix}$  is an eigenvector of  $\mathbf{A}$ , and where  $\mathbf{U}$  is another eigenvector that you would need to determine.*

Solution: