Section exam 1 for M341 (55060) Spring 2021

Released: Sunday March 7.

Due: 5pm, Thursday March 11. This is a strict deadline. Please allow yourself a margin!

Submission logistics: Submit through GradeScope. Please ensure that you know how this works well before the deadline in case difficulties arise.

Rules:

- This is an open book exam.
- The exam should be worked individually. Unlike the homeworks, you are *not* allowed to collaborate.
- You are allowed to use calculators, computers, etc, if you find them helpful. None of the questions should require extensive calculations. For the questions where motivations are required, you should at a minimum describe the steps that you took to compute the answer.
- Motivate your work unless a question specifically states that you do not have to.
- Write your answer inside the box given. This is important for GradeScope to be able to correctly scan your exam.

Question 1: (16p) In each of the four questions below, you are given the extended coefficient matrix $[A|B]$ for a linear system $AX = B$. Write down the full set of solutions to the linear system.

Note: For each system, you are also given a row equivalent system, as a help.

No motivation is required.

(a)

$$\left[\begin{array}{ccc|c} -2 & -2 & -5 & -4 \\ -4 & -1 & -3 & -12 \\ 2 & 2 & 4 & 2 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 6 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_3 = 2$

$x_2 + 3x_3 = 0 \Rightarrow x_2 = -6$

$2x_1 = 6 \Rightarrow x_1 = 3$

The solution set is:

$X_1 = 3 \qquad x_2 = -6 \qquad x_3 = 2$

(b)

$$\left[\begin{array}{ccc|c} 0 & 3 & 6 & 4 \\ -1 & 2 & 1 & 1 \\ 2 & 1 & 8 & 7 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The solution set is:

No sol^{ns}

(c)

$$\left[\begin{array}{ccc|c} 0 & 3 & 6 & 6 \\ -1 & 2 & 1 & 2 \\ 2 & 1 & 8 & 6 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 4 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution set is:

$X_1 = 2 - 3t$

$x_2 = 2 - 2t$

$x_3 = t$

t is a free variable

(d)

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ -2 & 4 & -1 & -1 & 0 \\ 3 & -6 & 1 & 1 & -1 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution set is:

$X_1 = -1 + 2t$

$x_2 = t$

$x_3 = 2 - s$

$x_4 = s$

s and t are free variables

Question 2: (14p) Consider the linear system

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 8 \\ -2 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ -9 \end{bmatrix}.$$

The solution to the system is $x_1 = 3$, $x_2 = -1$, $x_3 = 1$. In the box below, describe how you would solve the system using Gaussian elimination. Indicate clearly at each step which row operation that you perform. Use the notation based on extended coefficient matrices.

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 3 & 3 & 8 & 14 \\ -2 & -1 & -4 & -9 \end{array}\right] \sim \qquad R_2 \leftarrow R_2 - 3R_1, \quad R_3 \leftarrow R_3 + 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 \end{array}\right] \sim \qquad R_2 \Leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 2 \end{array}\right] \sim \qquad R_3 \leftarrow \frac{1}{2} R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \qquad R_1 \leftarrow R_1 - 2R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \qquad R_1 \leftarrow R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Hint: Always avoid dividing by quantities that may be zero!

Question 3: (15p) Let t be a real number, and consider the linear system

$$x_1 + x_3 = 1$$
$$x_1 + (1-t)x_2 = t - 2$$
$$-x_1 + (1-t)x_2 + (t-1)x_3 = t$$

Hint: You can solve this system by forming the extended coefficient matrix, and then driving it to REF form through Gaussian elimination. You only need three or four EROs.

(a) Specify an REF of the extended linear system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1-t & 0 & t-2 \\ -1 & 1-t & t-1 & t \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1-t & -1 & t-3 \\ 0 & 1-t & t & t+1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1-t & -1 & t-3 \\ 0 & 0 & t+1 & 4 \end{array}\right]$$

(b) Specify all values of t (if any) for which there is no solution:

$\boxed{t = -1}$ In this case $[A|B] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 0 & 4 \end{array}\right]$

inconsistent

(c) Specify all values of t (if any) for which there is a unique solution.
In this case, also specify the solution.

$\boxed{t \neq -1 \text{ and } t \neq 1}$

$$x_1 = 1 - x_3 = 1 - \frac{4}{1+t} = \frac{t-3}{t+1}$$

$$x_2 = \frac{1}{(1-t)}\left(t - 3 + x_3\right) = \frac{1}{(1-t)}\left(t-3+\frac{4}{1+t}\right) = \cdots = \frac{1-t}{1+t}$$

$$x_3 = \frac{4}{1+t}$$

(d) Specify all values of t (if any) for which there are infinitely many solutions.
In this case, also specify the solution set.

$\boxed{t=1}$ In this case $[A|B] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 = -1$

$x_2 = t$ ← free variable

$x_3 = 2$

If you get the answer right, you get full points for just specifying the values of t and the corresponding solutions. If you get it wrong, then a *brief* indication of your methodology may give partial credit.

Question 4: (15p) Consider the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ -2 & -1 & 4 \end{bmatrix}$.

(a) Provide a factorization $A = E_n E_{n-1} \cdots E_2 E_1$ where each E_j is a 3×3 matrix corresponding to an elementary row operation.

(b) Provide a factorization $A^{-1} = F_m F_{m-1} \cdots F_2 F_1$ where each F_j is a 3×3 matrix corresponding to an elementary row operation.

Hint: To solve this problem, consider the technique for computing the inverse of a matrix that we described in Lecture 10. Once you have worked out the answer to (a), almost no additional work is required for (b).

The idea is to drive A to the identity matrix through ERO's

$$F_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow F_1 A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & -1 & 4 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow F_2 F_1 A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow F_3 F_2 F_1 A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$F_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \Rightarrow F_4 F_3 F_2 F_1 A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow F_5 F_4 F_3 F_2 F_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \quad (*)$$

(a) $(*) \Rightarrow A = F_1^{-1} F_2^{-1} F_3^{-1} F_4^{-1} F_5^{-1}$

So
$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=E_5} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{=E_4} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{=E_3} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{=E_2} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=E_1}$$

(b) $(*) \Rightarrow A^{-1} = F_5 F_4 F_3 F_2 F_1$

Question 5: (15p) Let n be a positive integer, and let A, B, and C all be matrices of size $n \times n$. Suppose that A, B, C, and $A+B$ are all invertible matrices. Which of the following rules are then *necessarily* true:

(a) $A^{-1} - B^{-1} = B^{-1}(B-A)A^{-1}$.
(b) $A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}$.
(c) $(A+B)^{-1} = (I - A^{-1}B)^{-1}A^{-1}$.
(d) $(A+B)^{-1} = A^{-1}(I - A^{-1}B)^{-1}$.
(e) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

I typed (c) and (d) wrong. I had intended:

(c) $(A+B)^{-1} = (I + A^{-1}B)^{-1}A^{-1}$ TRUE

(d) $(A+B)^{-1} = A^{-1}(I + A^{-1}B)^{-1}$ FALSE

Circle TRUE or FALSE, and briefly motivate your answers.

(a) TRUE / FALSE

$B^{-1}(B-A)A^{-1} = B^{-1}BA^{-1} - B^{-1}AA^{-1} = A^{-1} - B^{-1}$

(b) TRUE / FALSE

$A^{-1}(B-A)B^{-1} = A^{-1}BB^{-1} - A^{-1}AB^{-1} = A^{-1} - B^{-1}$

(c) TRUE / FALSE

Set $A = I$ $B = I$

Then $(A+B)^{-1} = (2I)^{-1} = \frac{1}{2}I$

$A^{-1}(I - A^{-1}B)^{-1} = I^{-1}(I-I)^{-1}$ does not exist.

(d) TRUE / FALSE

Set $A = B = I$ again, for instance.

(e) TRUE / FALSE

$(ABC)^{-1} = ((AB)C)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$

Question 6: (15p) Let t be a real number, and consider the matrix

$$A = \begin{bmatrix} t & t \\ -t & t \end{bmatrix}.$$

Prove that there exists a value of t for which

$$\|A\mathbf{x}\| = \|\mathbf{x}\|$$

for every vector $\mathbf{x} \in \mathbb{R}^2$.

The correct value of t:

Proof:

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ denote an arbitrary vector. Then

$$Ax = \begin{bmatrix} t & t \\ -t & t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} tx_1 + tx_2 \\ -tx_1 + tx_2 \end{bmatrix} = t\begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix}$$

We find that

$$\|Ax\|^2 = (t(x_1+x_2))^2 + (t(-x_1+x_2))^2 =$$

$$= t^2[(x_1+x_2)^2 + (-x_1+x_2)^2] =$$

$$= t^2[x_1^2 + 2x_1x_2 + x_2^2 + x_1^2 - 2x_1x_2 + x_2^2] =$$

$$= t^2[2x_1^2 + 2x_2^2] =$$

$$= 2t^2\|x\|^2$$

So $\|Ax\| = \sqrt{2t^2}\,\|x\|$

Picking $t = \frac{1}{\sqrt{2}}$, we get $\|Ax\| = \|x\|$

(The value $t = -1/\sqrt{2}$ also works.)

Question 7: (5p) *[Note: This question is worth only five points!]* Consider the matrix

$$T = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For a positive integer n, define the numbers a_n, b_n, and c_n, as the first three entries in the top row of T^n. In other words

$$T^n = \begin{bmatrix} a_n & b_n & c_n & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}.$$

Work out what these numbers are, and provide an induction proof of your answer for c_n.

$a_n = 1$ $\qquad$ $b_n = n$ $\qquad$ $c_n = \frac{(n-1)n}{2}$

Proof of the claim for c_n: Since $T^n = T^{n-1}T$, we find that a_n, b_n, c_n satisfy:

$a_n = a_{n-1}$

$b_n = a_{n-1} + b_{n-1}$

$c_n = b_{n-1} + c_{n-1}$

The initial conditions are: $a_1 = 1$ $\quad b_1 = 1$ $\quad c_1 = 0$

We can now determine the sequences:

$\boxed{a_n}$ $\quad a_1 = 1$ and $a_n = a_{n-1}$ $\Rightarrow$ $a_n = 1$ for all n

$\boxed{b_n}$ $\quad b_n = 1 + b_{n-1}$ and $b_1 = 1$ immediately imply $b_n = n$

$\boxed{c_n}$ $\quad c_n = (n-1) + c_{n-1}$ and $c_1 = 0$.

We find $c_2 = 1+0 = 1$ $\quad c_3 = 2+1 = 3$ $\quad c_4 = 3+3 = 6$ $\quad c_5 = 4+6 = 10$ $\quad c_6 = 5+10 = 15$

$\boxed{\text{HYPOTHESIS: } \quad c_n = \frac{n(n-1)}{2}}$

Obviously true for $n=1$.

Assume true for $n-1$. Then

$c_n = (n-1) + \frac{(n-1)(n-2)}{2} = \frac{(n-1)}{2}(2+n-2) = \frac{(n-1)n}{2}$.

Question 8: (5p) Write a minimum of three sentences in the box below. Ideally, they would contain feedback on how you feel the class is going. Both positive and negative comments are very welcome! If you prefer to not provide feedback, then write anything you like. Any three full sentences will yield a full 5p.