

## DETERMINANTS

CLAIM: There exists a unique function  $f: M_{n,n} \rightarrow \mathbb{R}$  such that for every  $A \in M_{n,n}$ :

- (i) If  $E$  is the ERO that scales row  $i$  by the factor  $c$ , then  $f(EA) = cf(A)$ .  
This property holds even when  $c=0$   
(even though  $E$  is not an ERO in this case.)
- (ii) If  $E$  is the ERO that adds a scalar multiple of one row to another ( $\langle i \rangle \leftrightarrow \langle i \rangle + c\langle j \rangle$ ), then  $f(EA) = f(A)$ .
- (iii) If  $E$  is the ERO that swaps two rows ( $\langle i \rangle \leftrightarrow \langle j \rangle$ ), then  $f(EA) = -f(A)$
- (iv)  $f(I) = 1$

DEFN The function whose existence is asserted by the claim is called the DETERMINANT of the matrix. Given an  $n \times n$  matrix  $A$ , we write  $f(A) = \det(A)$ .

Alternatively, one sometimes writes  $f(A) = |A|$ .

Note that  $f(A)$  is a real number; it may be positive or negative, or zero.

NOTE The determinant cannot be defined for a rectangular matrix.

The properties in the claim are sufficient for computing  $\det(A)$  for any given matrix  $A$ .

Example Compute  $\det(A)$  for  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix}$ .

The idea is to use ERO's to turn  $A$  to  $I$ .

$$\det(A) = 2 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix} = -2 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

↑ scale row 1 by 2      ↗ swap rows 1 and 2

$$= -2 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = -2 \times 1 = -2$$

↑ subtract  $5 \times$  row 1 from row 2.

Answer:  $\det(A) = -2$ .

LEMMA If  $A$  has a row consisting of all zeros, then  $\det(A) = 0$ .

PROOF Let  $A_i$  be row  $i$  of  $A$  so  $A = \begin{bmatrix} \vdots & A_1 & \vdots \\ -A_1 & \vdots & - \\ \vdots & A_n & \vdots \\ -A_n & \vdots & - \end{bmatrix}$

Say row  $j$  is zero, so  $A_j = 0$ .

$$\text{Then } \det(A) = \det \begin{bmatrix} \vdots & A_1 & \vdots \\ -A_1 & \vdots & - \\ \vdots & A_j & \vdots \\ 0 & \vdots & 0 \\ -A_n & \vdots & - \end{bmatrix} = \det \begin{bmatrix} \vdots & A_1 & \vdots \\ -A_1 & \vdots & - \\ \vdots & 0 & \vdots \\ 0 & \vdots & 0 \\ -A_n & \vdots & - \end{bmatrix} =$$

↓ use property (i) for  $c=0$ .

$$= 0 \times \det \begin{bmatrix} \vdots & A_1 & \vdots \\ -A_1 & \vdots & - \\ \vdots & 0 & \vdots \\ 0 & \vdots & 0 \\ -A_n & \vdots & - \end{bmatrix} = 0.$$

THM Let  $A$  be an  $n \times n$  matrix. Then:

$$A \text{ is singular} \Leftrightarrow \det(A) = 0$$

PROOF

Case 1 - Suppose  $A$  is nonsingular

There exist ERO's  $\{\bar{E}_j\}_{j=1}^n$  s.t.

$$I = \bar{E}_n \bar{E}_{n-1} \cdots \bar{E}_2 \bar{E}_1 A \quad (\rightarrow \text{Why?})$$

Let  $c_j$  be the real numbers s.t.

$$\det(\bar{E}_j B) = c_j \det(B).$$

In other words,

$$c_j = \begin{cases} c & \text{if } \bar{E}_j \text{ is "type 1"} \\ 1 & \text{if } \bar{E}_j \text{ is "type 2"} \\ -1 & \text{if } \bar{E}_j \text{ is "type 3"} \end{cases}$$

Note:  $c_j \neq 0$ !

$Eg^n$  (\*) implies

$$\begin{aligned} 1 &= \det(I) = \det(\bar{E}_n \bar{E}_{n-1} \cdots \bar{E}_2 \bar{E}_1 A) = \\ &= c_n \det(\bar{E}_{n-1} \cdots \bar{E}_2 \bar{E}_1 A) = \\ &= c_n c_{n-1} \det(\bar{E}_{n-2} \cdots \bar{E}_1 A) = \\ &= \cdots = c_n c_{n-1} \cdots c_2 c_1 \det(A) \end{aligned}$$

$$\text{So } \det(A) = \frac{1}{c_1 c_2 \cdots c_n} \neq 0.$$

## Case 2 - Suppose $A$ is singular

Let  $B$  be the RREF of  $A$ .

Then  $\exists$  ERO's  $\{E_j\}$  s.t.

$$B = E_n E_{n-1} \dots E_1 A$$

As in case 1:

$$\underbrace{\det(B)}_{=0} = \underbrace{c_1 c_2 \dots c_n}_{\neq 0} \det(A)$$

Now  $\det(B)=0$  since the bottom row of  $B$  must be zero. (why?)

LEMMA Let  $A$  and  $B$  be  $n \times n$  matrices.

If  $AB$  is nonsingular, then both  $A$  and  $B$  are nonsingular.

PROOF Suppose  $AB$  is nonsingular. Set  $C = (AB)^{-1}$ .

Then  $ABC = I$ , so  $BC$  is an inverse of  $A$ .

Also:  $CAB = I$ , so  $CA$  is an inverse of  $B$ .

THM Let  $A$  and  $B$  be  $n \times n$  matrices. Then  
 $\det(AB) = \det(A)\det(B)$

PROOF

Case 1 - suppose  $A$  is singular

$AB$  must be singular due to the lemma we proved.

So  $\det(AB) = 0$  and  $\det(A) = 0$ .

The claim in the thm is true in this case.

Case 2 - suppose  $B$  is singular

The same as Case 1.

Case 3 - both  $A$  and  $B$  are non-sing.

There exist ERO's  $\{E_i\}_{i=1}^m$  and  $\{F_j\}_{j=1}^n$  s.t.

$$A = E_1 E_2 \cdots E_m$$

$$B = F_1 F_2 \cdots F_n$$

Let  $c_i$  be the scalar associated with  $E_i$ .

Let  $d_j$  — || —  $F_j$ .

$$\text{Then } \det(AB) = \det(E_1 E_2 \cdots E_m F_1 F_2 \cdots F_n) =$$

$$= \underbrace{c_1 c_2 \cdots c_m}_{=\det(A)} \underbrace{d_1 d_2 \cdots d_n}_{=\det(B)} \underbrace{\det(I)}_{=1}$$

## DETERMINANTS OF ELEMENTARY MATRICES:

Let us consider some  $3 \times 3$  matrices associated with ERO's.

"Type 1" - scale row 2 by  $c$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E^t = E$$

$$\det(E) = c \quad \det(E^{-1}) = \frac{1}{c} \quad \det(E^t) = \det(E)$$

"Type 2" - add  $c \times$  row 1 to row 3

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix} \quad E^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(E) = 1 \quad \det(E^{-1}) = 1 \quad \det(E^t) = 1$$

"Type 3" - swap rows 1 and 2

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E^{-1} = E \quad E^t = E$$

$$\det(E) = -1 \quad \det(E^{-1}) = -1 \quad \det(E^t) = -1$$

THM Let  $A$  be a nonsing. matrix.

$$\text{Then } \det(A^{-1}) = \frac{1}{\det(A)}$$

PROOF  $I = AA^{-1}$

$$\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

$= 1$

THM  $\det(A^t) = \det(A)$

PROOF If  $A$  is sing, then so is  $A^t$ ,  
so  $\det(A^t) = 0 = \det(A)$ .

If  $A$  is nonsing, then

$$A = E_1 E_2 \cdots E_n$$

for some ERO's  $E_j$ . Then

$$\begin{aligned}\det(A^t) &= \det(E_n^t E_{n-1}^t \cdots E_1^t) = \\ &= \det(E_n^t) \det(E_{n-1}^t) \cdots \det(E_1^t) = \\ &= C_n C_{n-1} \cdots C_2 C_1 = \\ &= C_1 C_2 \cdots C_{n-1} C_n = \\ &= \det(E_1) \det(E_2) \cdots \det(E_n) = \\ &= \det(E_1 E_2 \cdots E_n) = \det(A).\end{aligned}$$

THEM Suppose  $A$  is triangular (upper or lower).

Then  $\det(A) = \text{product of diagonal entries.}$

PROOF Let us consider a  $3 \times 3$  upper triangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

If any of the diagonal entries is zero,

then the RREF of  $A$  has a zero row at the bottom. (why?)

$$\text{So } \det(A) = 0 = a_{11}a_{22}a_{33}$$

Let us suppose all diagonal entries are nonzero.

$$\det(A) = a_{11} \det \begin{bmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} =$$

$$= a_{11}a_{22}a_{33} \det \begin{bmatrix} 1 & a_{12}/a_{11} & a_{13}/a_{11} \\ 0 & 1 & a_{23}/a_{22} \\ 0 & 0 & 1 \end{bmatrix} =$$

$= \left\{ \begin{array}{l} \text{Do ERO's of type 2 only!} \\ \text{These do not change the determinant} \end{array} \right\} =$

$$= a_{11}a_{22}a_{33} \det(I) = a_{11}a_{22}a_{33}$$

The generalization to  $n \times n$  matrices is obvious.

The proof for lower triangular matrices is analogous.

CONSEQUENCE: In order to compute the determinant, you can stop after driving the matrix to its REF.  
 No need for RREF.  
 Half the work! (Almost...)

THM  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

PROOF Case 1:  $c \neq 0$ .

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= c \det \begin{bmatrix} 1 & b/a \\ c & d \end{bmatrix} = \\ &= c \det \underbrace{\begin{bmatrix} 1 & b/c \\ 0 & d - \frac{bc}{a} \end{bmatrix}}_{\text{upper tri!}} = c \times 1 \times \left( d - \frac{bc}{a} \right) = cd - bc \end{aligned}$$

Case 2:  $c \neq 0$

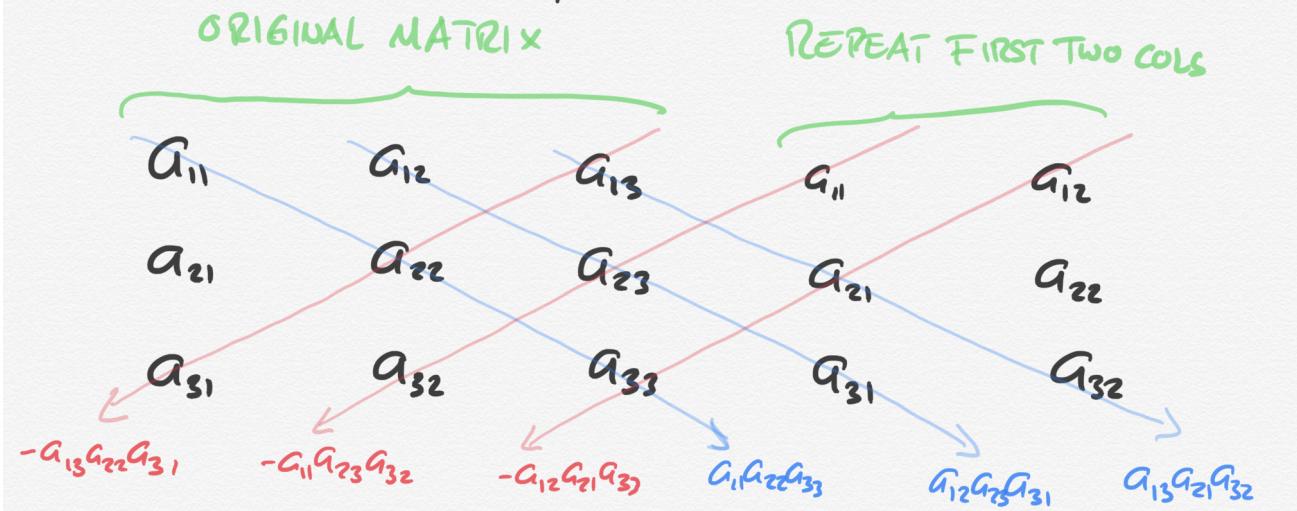
$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \xrightarrow{\text{swap the rows}} - \det \underbrace{\begin{bmatrix} c & d \\ 0 & b \end{bmatrix}}_{\text{upper tri!}} = -bc$$

Example  $\det \begin{bmatrix} 7 & 5 \\ 2 & 3 \end{bmatrix} = 7 \cdot 3 - 2 \cdot 5 = 21 - 10 = 11$

$$\text{THM} \quad \det \begin{Bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

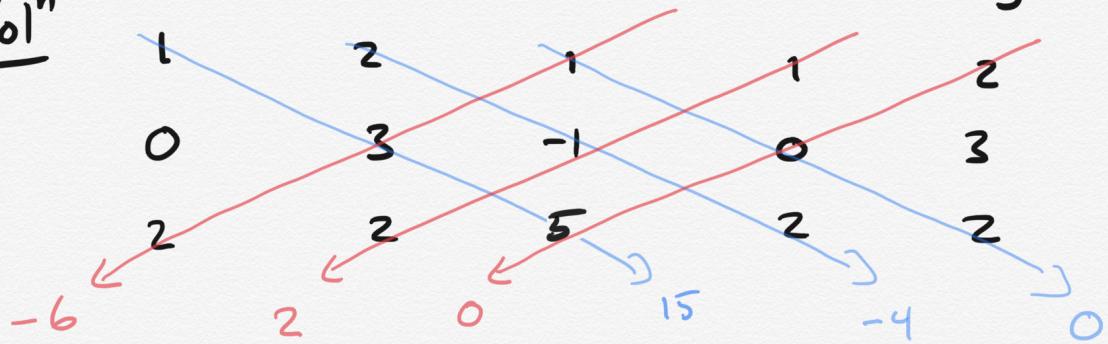
PROOF We skip...

There is a simple mnemonic to remember the rule:  
Write down a  $3 \times 5$  matrix with the first  
two columns repeated:



Example Compute the determinant of  $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & -1 \\ 2 & 2 & 5 \end{bmatrix}$

Sol<sup>n</sup>

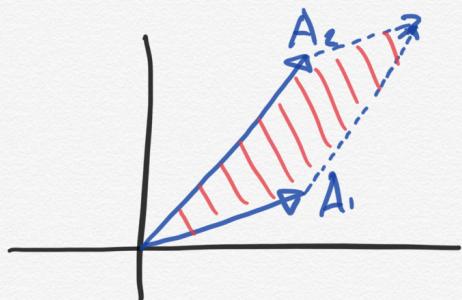


$$\det(A) = 15 - 4 + 0 - 6 + 2 + 0 = 7$$

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## GEOMETRIC INTERPRETATION OF DETERMINANT.

Let  $A = [A_1 \ A_2]$  be a  $2 \times 2$  matrix with columns  $A_1$  &  $A_2$ .



$|\det(A)| = \text{Area of parallelogram formed by the columns of } A.$

Since  $\det(A) = \det(A^T)$ , you could use the rows instead of columns too.

For a  $3 \times 3$  matrix, there is an analogous result.

$|\det(A)| = \text{Volume of parallelepiped formed by the three columns (or rows) of } A.$

Note that a matrix is singular iff this volume is zero, which means all three columns lie in a plane, or on the same line.