## Recall a theorem about the properties of vectors

Theorem 1.3 Let $\mathbf{x}=\left[x_{1}, x_{2}, \ldots, x_{n}\right], \mathbf{y}=\left[y_{1}, y_{2}, \ldots, y_{n}\right]$ and $\mathbf{z}=\left[z_{1}, z_{2}, \ldots, z_{n}\right]$ be any vectors in $\mathbb{R}^{n}$, and let $\boldsymbol{c}$ and $\boldsymbol{d}$ be any real numbers (scalars). Let $\mathbf{0}$ represent the zero vector in $\mathbb{R}^{n}$. Then
(1) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$
Commutative Law of Addition
(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$
Associative Law of Addition
(3) $\mathbf{0}+\mathbf{x}=\mathbf{x}+\mathbf{0}=\mathbf{x}$
Existence of Identity Element for Addition
(4) $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$
Existence of Inverse Elements for Addition
(5) $c(\mathbf{x}+\mathbf{y})=c \mathbf{x}+c \mathbf{y}$
Distributive Laws of Scalar Multiplication
(6) $(c+d) \mathbf{x}=c \mathbf{x}+d \mathbf{x}$
(7) $(c d) \mathbf{x}=c(d \mathbf{x})$ over Addition
(8) $1 \mathbf{x}=\mathbf{x}$
Associativity of Scalar Multiplication
Identity Property for Scalar Multiplication

## And a similar one about matrices

Theorem 1.12 Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ be $m \times n$ matrices (elements of $\mathcal{M}_{m n}$ ), and let $c$ and $d$ be scalars. Then
(1) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
(2) $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$

Commutative Law of Addition
(3) $\mathbf{O}_{m n}+\mathbf{A}=\mathbf{A}+\mathbf{O}_{m n}=\mathbf{A}$
(4) $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}_{m n}$

Associative Law of Addition
(5) $c(\mathbf{A}+\mathbf{B})=c \mathbf{A}+c \mathbf{B}$
(6) $(c+d) \mathbf{A}=c \mathbf{A}+d \mathbf{A}$
(7) $\quad(c d) \mathbf{A}=c(d \mathbf{A})$
(8) $\mathbf{1}(\mathbf{A})=\mathbf{A}$

Existence of Identity Element for Addition
Existence of Inverse Elements for Addition
Distributive Laws of Scalar
Multiplication over Addition
Associativity of Scalar Multiplication
Identity Property for Scalar Multiplication

Theorem 1.3 Let $\mathbf{x}=\left[x_{1}, x_{2}, \ldots, x_{n}\right], \mathbf{y}=\left[y_{1}, y_{2}, \ldots, y_{n}\right]$ and $\mathbf{z}=\left[z_{1}, z_{2}, \ldots, z_{n}\right]$ be any vectors in $\mathbb{R}^{n}$, and let $c$ and $d$ be any real numbers (scalars). Let $\mathbf{0}$ represent the zero vector in $\mathbb{R}^{n}$. Then

| (1) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ | Commutative Law of Addition |
| :--- | :--- |
| (2) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$ | Associative Law of Addition |
| (3) $\mathbf{0}+\mathbf{x}=\mathbf{x}+\mathbf{0}=\mathbf{x}$ | Existence of Identity Element for Addition |
| (4) $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$ | Existence of Inverse Elements for Addition |
| (5) $c(\mathbf{x}+\mathbf{y})=c \mathbf{x}+c \mathbf{y}$ | Distributive Laws of Scalar Multiplication |
| (6) $(c+d) \mathbf{x}=c \mathbf{x}+d \mathbf{x}$ | over Addition |
| (7) $(c d) \mathbf{x}=c(d \mathbf{x})$ | Associativity of Scalar Multiplication |
| (8) $1 \mathbf{x}=\mathbf{x}$ | Identity Property for Scalar Multiplication |

Theorem 1.12 Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ be $m \times n$ matrices (elements of $\mathcal{M}_{m n}$ ), and let $\boldsymbol{c}$ and $d$ be scalars. Then

| (1) | $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ | Commutative Law of Addition |
| :--- | :--- | :--- |
| (2) | $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$ | Associative Law of Addition |
| (3) | $\mathbf{O}_{m n}+\mathbf{A}=\mathbf{A}+\mathbf{O}_{m n}=\mathbf{A}$ | Existence of Identity Element for Addition |
| (4) | $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{\mathbf { O } _ { m n }}$ | Existence of Inverse Elements for Addition |
| (5) | $c(\mathbf{A}+\mathbf{B})=c \mathbf{A}+c \mathbf{B}$ | Distributive Laws of Scalar |
| (6) | $(c+d) \mathbf{A}=c \mathbf{A}+d \mathbf{A}$ | Multiplication over Addition |
| (7) | $(c d) \mathbf{A}=c(d \mathbf{A})$ | Associativity of Scalar Multiplication |
| (8) | $\mathbf{1 ( A )})=\mathbf{A}$ | Identity Property for Scalar Multiplication |

Theorem 1.3 Let $\mathbf{x}=\left[x_{1}, x_{2}, \ldots, x_{n}\right], \mathbf{y}=\left[y_{1}, y_{2}, \ldots, y_{n}\right]$ and $\mathbf{z}=\left[z_{1}, z_{2}, \ldots, z_{n}\right]$ be any vectors in $\mathbb{R}^{n}$, and let $\boldsymbol{c}$ and $\boldsymbol{d}$ be any real numbers (scalars). Let $\mathbf{0}$ represent the zero vector in $\mathbb{R}^{n}$. Then

| (1) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ | Commutative Law of Addition |
| :--- | :--- |
| (2) $\mathbf{x}+(\mathbf{y}+\mathbf{z})=(\mathbf{x}+\mathbf{y})+\mathbf{z}$ | Associative Law of Addition |
| (3) $\mathbf{0}+\mathbf{x}=\mathbf{x}+\mathbf{0}=\mathbf{x}$ | Existence of Identity Element for Addition |
| (4) $\mathbf{x}+(-\mathbf{x})=(-\mathbf{x})+\mathbf{x}=\mathbf{0}$ | Existence of Inverse Elements for Addition |
| (5) $c(\mathbf{x}+\mathbf{y})=c \mathbf{x}+c \mathbf{y}$ | Distributive Laws of Scalar Multiplication |
| (6) $(c+d) \mathbf{x}=c \mathbf{x}+d \mathbf{x}$ | over Addition |
| (7) $(c d) \mathbf{x}=c(d \mathbf{x})$ | Associativity of Scalar Multiplication |
| (8) $1 \mathbf{x}=\mathbf{x}$ | Identity Property for Scalar Multiplication |

Theorem 1.12 Let $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ be $m \times n$ matrices (elements of $\mathcal{M}_{m n}$ ), and let $\boldsymbol{c}$ and $\boldsymbol{d}$ be scalars. Then
(1) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \quad$ Commutative Law of Addition
(2) $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C} \quad$ Associative Law of Addition
(3) $\mathbf{O}_{m n}+\mathbf{A}=\mathbf{A}+\mathbf{O}_{m n}=\mathbf{A} \quad$ Existence of Identity Element for Addition
(4) $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}_{m n} \quad$ Existence of Inverse Elements for Addition
(5) $c(\mathbf{A}+\mathbf{B})=c \mathbf{A}+c \mathbf{B} \quad$ Distributive Laws of Scalar
(6) $(c+d) \mathbf{A}=c \mathbf{A}+d \mathbf{A}$

Multiplication over Addition
(7) $\quad(c d) \mathbf{A}=c(d \mathbf{A})$

Associativity of Scalar Multiplication
(8) $\mathbf{1}(\mathbf{A})=\mathbf{A} \quad$ Identity Property for Scalar Multiplication

## Similar theorems hold about many different objects:

- $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ could be polynomials.
- $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ could be continuous functions on, say, the interval $[-1,1]$.
- $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ could be geometric vectors that all lie in a plane $L$ through the origin.

Definition A vector space is a set $\mathcal{V}$ together with an operation called vector addition (a rule for adding two elements of $\mathcal{V}$ to obtain a third element of $\mathcal{V}$ ) and another operation called scalar multiplication (a rule for multiplying a real number times an element of $\mathcal{V}$ to obtain a second element of $\mathcal{V}$ ) on which the following ten properties hold:

For every $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in $\mathcal{V}$, and for every $a$ and $b$ in $\mathbb{R}$,

| (A) | $\mathbf{u}+\mathbf{v} \in \mathcal{V}$ | Closure Property of Addition |
| :---: | :---: | :---: |
| (B) | $a \mathbf{u} \in \mathcal{V}$ | Closure Property of Scalar Multiplication |
| (1) | $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ | Commutative Law of Addition |
| (2) | $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ | Associative Law of Addition |
| (3) | There is an element $\mathbf{0}$ of $\mathcal{V}$ so that for every $y$ in $\mathcal{V}$ we have $\mathbf{0}+\mathbf{y}=\mathbf{y}=\mathbf{y}+\mathbf{0}$ | Existence of Identity Element for Addition |
| (4) | There is an element $-\mathbf{u}$ in $\mathcal{V}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}=(-\mathbf{u})+\mathbf{u}$. | Existence of Additive Inverse |
| (5) | $a(\mathbf{u}+\mathbf{v})=(a \mathbf{u})+(a \mathbf{v})$ | Distributive Laws for Scalar |
| (6) | $(a+b) \mathbf{u}=(a \mathbf{u})+(b \mathbf{u})$ | Multiplication over Addition |
| (7) | $(a b) \mathbf{u}=a(b \mathbf{u})$ | Associativity of Scalar Multiplication |
| (8) | $1 \mathbf{u}=\mathbf{u}$ | Identity Property for Scalar Multiplication |

The elements of a vector space $\mathcal{V}$ are called vectors.

Theorem 4.1 Let $\mathcal{V}$ be a vector space. Then, for every vector $\mathbf{v}$ in $\mathcal{V}$ and every real number $a$, we have
(1) $a 0=0$

Any scalar multiple of the zero vector yields the zero vector.
(2) $0 \mathbf{v}=\mathbf{0}$

The scalar zero multiplied by any vector yields the zero vector.
(3) $(-1) \mathbf{v}=-\mathbf{v}$

The scalar - 1 multiplied by any vector yields the additive inverse of that vector.
(4) If $a \mathbf{v}=\mathbf{0}$, then $a=0$ or $\mathbf{v}=\mathbf{0}$

If a scalar multiplication yields the zero vector, then either the scalar is zero, or the vector is the zero vector, or both.

