Final exam for M341 (55060) Spring 2021 — SOLUTIONS

Released: Saturday May 8, 2021.

Due: 5pm on Wednesday May 12, 2021.

Submission logistics: Submit through GradeScope. Please ensure that you know how this works well before the deadline in case difficulties arise.

Rules:

- This is an open book exam.
- The exam should be worked individually. Unlike the homeworks, you are *not* allowed to collaborate.
- You are allowed to use calculators, computers, etc, if you find them helpful. None of the questions should require extensive calculations. For the questions where motivations are required, you should at a minimum describe the steps that you took to compute the answer. For example, if you did row eliminations, then specify the matrix you start with, and the matrix that you end up with.
- Motivate your work unless a question specifically states that you do not have to.
- Write your answer inside the box given. This is important for GradeScope to be able to correctly scan your exam.

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Question 1: (18p) Let s and t be real numbers, and consider the linear system

$$(1)\qquad \left\{\begin{array}{rrrcr} x_1 & +t\,x_2 & +(3+t)\,x_3 & = & 2, \\ -x_1 & -2x_2 & -x_3 & = & s-5, \\ x_1 & +(2t-2)x_2 & +(6+2t)x_3 & = & s, \end{array}\right.$$

where x_1, x_2, x_3 are the three unknown variables.

(a) For which values of s and t, if any, does the system (1) have a unique solution? Specify the solution set if it exists.
(b) For which values of s and t, if any, does the system (1) not have any solution?
(c) For which values of s and t, if any, does the system (1) have infinitely many solutions? Specify the solution set if it exists.

Solution: First we calculate an REF of the extended coefficient matrix

$$\left[\begin{array}{rrr|r} 1 & t & 3+t & 2 \\ -1 & -2 & -1 & s-5 \\ 1 & 2t+2 & 6+2t & s \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & t & 3+t & 2 \\ 0 & t-2 & 2+t & s-3 \\ 0 & t-2 & 3+t & s-2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & t & 3+t & 2 \\ 0 & t-2 & 2+t & s-3 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

We see that the important distinction is whether $t = 2$ or not. If $t = 2$, we get:

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 2 \\ 0 & 0 & 4 & s-3 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

We can now easily tell the different situations apart:

(a) $\boxed{\text{The system has a unique solution if } t \neq 2.}$ We find

$$x_3 = 1, \qquad x_2 = \frac{s-t-5}{t-2}, \qquad x_1 = -1 - t - t\frac{s-t-5}{t-2}.$$

(b) $\boxed{\text{The system has no solution if } t = 2 \text{ and } s \neq 7.}$

(c) $\boxed{\text{The system has infinitely many solutions if } t = 2 \text{ and } s = 7.}$ The RREF becomes:

$$\left[\begin{array}{rrr|r} 1 & 2 & 5 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So the solution set is

$$x_3 = 1, \qquad x_2 = t, \qquad x_3 = -3 - 2t,$$

where t is an arbitrary real number.

Question 2: (20p) Let $\mathbf{A}$ be a 3×3 matrix with rows $\{\mathbf{A}_i\}_{i=1}^3$ so that $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}$.

You know that $\det(\mathbf{A}) = 3$. For each matrix given below, specify the value of its determinant in the cases where you have enough information to evaluate it. Please motivate each answer briefly.

(a) $\det\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 + 2\mathbf{A}_1 \\ 5\mathbf{A}_3 \end{bmatrix}\right)$

(b) $\det(\mathbf{A} + \mathbf{I}) =$

(c) $\det(\mathbf{A} + \mathbf{A}) =$

(d) $\det\left(\mathbf{A}^{-1}\right) =$

(e) $\det\left(\begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{A}_1 & \mathbf{A}_1 \cdot \mathbf{A}_2 & 2\mathbf{A}_1 \cdot \mathbf{A}_3 \\ \mathbf{A}_2 \cdot \mathbf{A}_1 & \mathbf{A}_2 \cdot \mathbf{A}_2 & 2\mathbf{A}_2 \cdot \mathbf{A}_3 \\ 2\mathbf{A}_3 \cdot \mathbf{A}_1 & 2\mathbf{A}_3 \cdot \mathbf{A}_2 & 4\mathbf{A}_3 \cdot \mathbf{A}_3 \end{bmatrix}\right) =$

Solution:

(a) $\det\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 + 2\mathbf{A}_1 \\ 5\mathbf{A}_3 \end{bmatrix}\right) \overset{(1)}{=} \det\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ 5\mathbf{A}_3 \end{bmatrix}\right) \overset{(2)}{=} 5\det\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}\right) = 5 \cdot 3 = 15.$

Step (1) is a type 2 ERO. Step (2) is a type 3 ERO.

(b) There is simply not enough information to work this out. Consider

$$\det\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \mathbf{I}\right) = \det\left(\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$$

and

$$\det\left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mathbf{I}\right) = \det\left(\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 4.$$

(c) $\det(\mathbf{A} + \mathbf{A}) = \det(2\mathbf{A}) = 2^3 \det(\mathbf{A}) = 8 \cdot 3 = 24.$

(d) $\det\left(\mathbf{A}^{-1}\right) = \det(\mathbf{A})^{-1} = 3^{-1} = 1/3.$

(e) Set $\mathbf{B} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ 2\mathbf{A}_3 \end{bmatrix}$. Then

$$\det\left(\begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{A}_1 & \mathbf{A}_1 \cdot \mathbf{A}_2 & 2\mathbf{A}_1 \cdot \mathbf{A}_3 \\ \mathbf{A}_2 \cdot \mathbf{A}_1 & \mathbf{A}_2 \cdot \mathbf{A}_2 & 2\mathbf{A}_2 \cdot \mathbf{A}_3 \\ 2\mathbf{A}_3 \cdot \mathbf{A}_1 & 2\mathbf{A}_3 \cdot \mathbf{A}_2 & 4\mathbf{A}_3 \cdot \mathbf{A}_3 \end{bmatrix}\right) = \det\left(\mathbf{B}\mathbf{B}^{\mathrm{T}}\right) = \det(\mathbf{B}) \det\left(\mathbf{B}^{\mathrm{T}}\right) = \det(\mathbf{B})^2 = (2\det(\mathbf{A}))^2 = (2 \cdot 3)^2 = 36.$$

Question 3: (18p) Let $\mathbf{A} = \begin{bmatrix} 0 & -2 & 3 & 3 \\ -2 & 0 & 3 & 3 \\ 3 & 3 & 0 & -2 \\ 3 & 3 & -2 & 0 \end{bmatrix}$, let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and let $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. Observe that $\mathbf{A}$ is symmetric. You know that $\mathbf{A}$ has exactly three distinct eigenvalues, and that the vectors $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of $\mathbf{A}$. Specify all eigenvalues of $\mathbf{A}$, and provide bases for the corresponding eigenspaces.

Solution: We first determine the eigenvalues and eigenspaces associated with $\mathbf{u}$ and $\mathbf{v}$, since this is very straight-forward.

$\mathbf{Au} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = 4\mathbf{u}$. We see that $\lambda_1 = 4$. We then determine the corresponding eigenspace:

$$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -4 & -2 & 3 & 3 \\ -2 & -4 & 3 & 3 \\ 3 & 3 & -4 & -2 \\ 3 & 3 & -2 & -4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $E_1 = \operatorname{span}(\mathbf{u})$ since there is only one free variable.

$\mathbf{Av} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} = 2\mathbf{v}$. We see that $\lambda_2 = 2$. We then determine the corresponding eigenspace:

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -2 & -2 & 3 & 3 \\ -2 & -2 & 3 & 3 \\ 3 & 3 & -2 & -2 \\ 3 & 3 & -2 & -2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We now find that there is an additional linearly independent eigenvector, for instance $\mathbf{w} = [0, 0, 1, -1]^{\mathrm{T}}$. So $E_2 = \operatorname{span}(\mathbf{v}, \mathbf{w})$.

Now comes the slightly trickier question: How do we find the third eigenvalue and the fourth eigenvector? Well, recall that since $\mathbf{A}$ is symmetric, we know that there is an orthogonal basis consisting of eigenvectors for $\mathbf{A}$. The elusive fourth eigenvector $\mathbf{q}$ must be orthogonal to the previous three ones, which means that

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{\mathrm{T}}\mathbf{q} \\ \mathbf{v}^{\mathrm{T}}\mathbf{q} \\ \mathbf{w}^{\mathrm{T}}\mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{u}^{\mathrm{T}} \\ \mathbf{v}^{\mathrm{T}} \\ \mathbf{w}^{\mathrm{T}} \end{bmatrix} \mathbf{q}.$$

Now

$$\begin{bmatrix} \mathbf{u}^{\mathrm{T}} \\ \mathbf{v}^{\mathrm{T}} \\ \mathbf{w}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

A solution is given by $\mathbf{q} = [1, 1, -1, 1]^{\mathrm{T}}$. It is easily verified that $\mathbf{Aq} = -8\mathbf{q}$ so $\lambda_3 = -8$ and $E_3 = \operatorname{span}(\mathbf{q})$.

Question 4a:(10p) Let $\mathbf{x} = [1, -2, 2]$ be a vector, and let $L = \{[t, 2t, t] : t \in \mathbb{R}\}$ be a line in $\mathbb{R}^3$. Compute the distance d between $\mathbf{x}$ and L, where $d = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in L\}$. Explain how you can use the notion of a *projection* of a vector (the function $\mathrm{proj}_{\mathbf{a}}(\mathbf{b})$ that we discussed in Lecture 3) to compute d.

Solution:

(a) $L = \mathrm{span}(\mathbf{u})$ where $\mathbf{u} = [1, 2, 1]$. We compute the orthogonal projection $\mathbf{y}$ of $\mathbf{x}$ onto L:

$$\mathbf{y} = \mathrm{proj}_{\mathbf{u}}(\mathbf{x}) = \frac{\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{u}\|^2}\,\mathbf{u} = \frac{-1}{6}[1, 2, 1].$$

We find that

$$d = \|\mathbf{x} - \mathbf{y}\| = \|\tfrac{1}{6}[7, -10, 13]\| = \tfrac{1}{6}\|[7, -10, 13]\| = \tfrac{1}{6}\sqrt{49 + 100 + 169} = \frac{\sqrt{318}}{6} = 2.97\cdots$$

Question 4b:(10p) Let $\mathbf{x} = [2, -1, -1]$ be a vector, and let $L = \mathrm{span}\big([1, 1, 1],\, [1, 2, 1]\big)$ be a plane through the origin in $\mathbb{R}^3$. Determine an orthogonal basis for L, and specify the distance d between $\mathbf{x}$ and L, where $d = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in L\}$.

Solution:

Set $\mathbf{u} = [1, 1, 1]$ and $\mathbf{v} = [1, 2, 1]$ so that $L = \mathrm{span}(\mathbf{u}, \mathbf{v})$. Next, let us take one step of Gram-Schmidt to find a new basis vector $\mathbf{w}$ so that $\{\mathbf{u}, \mathbf{w}\}$ forms an *orthogonal* basis for L. We first determine the direction:

$$\mathbf{w}' = \mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = [1, 2, 1] - \frac{4}{3}[1, 1, 1] = \frac{1}{3}[-1, 2, -1].$$

Rescaling to get easier numbers to work with (this is not a necessary step!), we set $\mathbf{w} = [-1, 2, -1]$. We can now easily compute the orthogonal projection $\mathbf{y}$ of $\mathbf{x}$ onto L:

$$\mathbf{y} = \mathrm{proj}_L(\mathbf{x}) = \frac{\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{u}\|^2}\,\mathbf{u} + \frac{\mathbf{w} \cdot \mathbf{x}}{\|\mathbf{w}\|^2}\,\mathbf{w} = 0 + \frac{-3}{6}[-1, 2, -1] = [1/2, -1, 1/2].$$

Finally

$$d = \|\mathbf{x} - \mathbf{y}\| = \|[2, -1, -1] - [1/2, -1, 1/2]\| = \|[3/2, 0, -3/2]\| = \sqrt{(3/2)^2 + (-3/2)^2} = \frac{3}{\sqrt{2}} = 2.12\cdots$$

Question 5a: (7p) (7p) Let $\mathbf{A}$ and $\mathbf{B}$ be a 4×4 matrices. You know that there are two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{Au} = \mathbf{Av}$. You also know that $\dim(\text{range}(\mathbf{AB})) = 3$. Do you have enough information to determine the *rank* of $\mathbf{A}$? Specify the rank if the answer is yes.

Solution: Set $\mathbf{w} = \mathbf{u} - \mathbf{v}$. We know that $\mathbf{w} \neq \mathbf{0}$ since $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Since $\mathbf{Aw} = \mathbf{Au} - \mathbf{Av} = \mathbf{0}$, we know that $\dim(\ker(\mathbf{A})) \geq 1$ and consequently.

$$\text{rank}(\mathbf{A}) = 4 - \dim(\ker(\mathbf{A})) \leq 4 - 1 = 3.$$

It is clear that $\text{range}(\mathbf{AB}) \subseteq \text{range}(\mathbf{A})$. Since $\dim(\text{range}(\mathbf{AB})) = 3$, it follows that

$$\text{rank}(\mathbf{A}) = \dim(\text{range}(\mathbf{A})) \geq 3.$$

Taken together, the two inequalities imply that $\dim(\text{range}(\mathbf{A})) = 3$.

Question 5b: (7p) Let L be a linear map from $\mathcal{P}_2$ to $\mathcal{P}_3$, and let K be a linear map from $\mathcal{P}_3$ to $\mathcal{P}_3$. Define the linear map M from $\mathcal{P}_2$ to $\mathcal{P}_3$ via $M(p) = K(L(p))$. You know that $\ker(L) = \{0\}$ and that $\dim(\text{range}(K)) = 4$. Is M one-to-one? Is M onto?

Solution: Since $\ker(L) = \{0\}$, we know the L is one-to-one.

We also know the K is one-to-one since $\dim(\ker(K)) = 4 - \dim(\text{range}(K)) = 4 - 4 = 0$.

Now suppose that $M(p) = K(L(p)) = 0$. Since K is one-to-one, we know that $L(p) = 0$. Since L is one-to-one, we know that $p = 0$. It follows that M is one-to-one

M maps a vectors space of dimension 3 to a vector space of dimension 4. This means that it is impossible for M to be onto.

Question 5c: (4p) Let V denote the linear space consisting of all functions f of the form

$$f(x) = a_0 + \sum_{j=1}^{5}\big(a_j\,\cos(jx) + b_j\,\sin(jx)\big), \tag{2}$$

where a_j and b_j are real numbers. Let t denote a real number, and consider the operator L defined by $g = Lf$ where $g(x) = f(x-t)$. For which values of t is L a linear operator from V to V? When is the operator L one-to-one?

Solution: For any x, t, and j, we have

$$\begin{aligned}\cos(j(x-t)) =&\ \cos(jx)\cos(jt) + \sin(jx)\sin(jt)\\ \sin(j(x-t)) =&\ \cos(jx)\sin(jt) - \sin(jx)\cos(jt).\end{aligned}$$

It follows that if f is any function of the form (2) and $g = Lf$, then

$$g(x) = f(x-t) = a_0 + \sum_{j=1}^{5}\big(a_j'\,\cos(jx) + b_j'\,\sin(jx)\big),$$

where

$$\begin{aligned}a_j' &= a_j\cos(jt) + b_j\sin(jt)\\ b_j' &= a_j\sin(jt) - b_j\cos(jt).\end{aligned}$$

So $g \in V$. It is straight-forward to verify that $L(c_1 f_1 + c_2 f_2) = c_1 L(f_1) + c_2 L(f_2)$ so L is indeed linear. $\boxed{L \text{ is a linear map from } V \text{ to } V \text{ for every } t.}$

It remains to check if L is one-to-one. Suppose $g = L(f) = 0$. This means that $g(x) = f(x-t) = 0$ for every x. But then we must have $f(x) = 0$ for every x too. So $f = 0$, and $\boxed{L \text{ is one-to-one for every } t.}$

Question 6: (6p) Let $\mathbf{A}$ be a matrix of size $m \times k$, and rank k. Set

$$\mathbf{B} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}.$$

In answering this question, you may without proof use that: (i) The matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is invertible. (ii) The matrix $\mathbf{A}$ can be written in the form $\mathbf{A} = \mathbf{UC}$ where $\mathbf{U}$ is an $m \times k$ matrix that satisfies $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$ and where $\mathbf{C}$ is an invertible matrix. (Both of these facts follow from the statement that $\mathbf{A}$ has rank k.)

(a) Prove that $\mathbf{BA} = \mathbf{I}$.
(b) Prove that $\mathbf{AB}$ is the *orthogonal projection* (as defined in Section 6.2) onto the column space of $\mathbf{A}$.

Solution: (a)

$$\mathbf{BA} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{A} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right) = \mathbf{I}$$

(b) Using the relation $\mathbf{A} = \mathbf{UC}$ we find that

$$\mathbf{AB} = \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}} = \mathbf{UC}\left(\mathbf{C}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}\mathbf{UC}\right)^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}}.$$

Use that $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$ and that $\mathbf{C}$ is invertible to find that

$$\mathbf{AB} = \mathbf{UC}\left(\mathbf{C}^{\mathrm{T}}\mathbf{C}\right)^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}} = \mathbf{UCC}^{-1}\left(\mathbf{C}^{\mathrm{T}}\right)^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}} = \mathbf{UU}^{\mathrm{T}}.$$

Let $\{\mathbf{u}_i\}_{i=1}^k$ denote the columns of $\mathbf{U}$. The set $\{\mathbf{u}_i\}_{i=1}^k$ forms a spanning set for the column space of $\mathbf{A}$ since if $\mathbf{y} = \mathbf{Ax}$, then $\mathbf{y} = \mathbf{Ux}'$ where $\mathbf{x}' = \mathbf{Cx}$.

Next, observe that $\{\mathbf{u}_i\}_{i=1}^k$ is an orthonormal set since $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$.

So $\{\mathbf{u}_i\}_{i=1}^k$ is an ON-basis for the column space of $\mathbf{A}$, and we find that for any $\mathbf{x} \in \mathbb{R}^m$ we have

$$\mathbf{ABx} = \mathbf{UU}^{\mathrm{T}}\mathbf{x} = \sum_{i=1}^{k}\left(\mathbf{u}_i \cdot \mathbf{x}\right)\mathbf{u}_i.$$

We recognize precisely the formula for orthogonal projection onto a set with ON basis $\{\mathbf{u}_i\}_{i=1}^k$.