Final exam for M341 (55060) Spring 2021

Released: Saturday May 8, 2021.

Due: 5pm on Wednesday May 12, 2021.

Submission logistics: Submit through GradeScope. Please ensure that you know how this works well before the deadline in case difficulties arise.

Rules:

- This is an open book exam.
- The exam should be worked individually. Unlike the homeworks, you are *not* allowed to collaborate.
- You are allowed to use calculators, computers, etc, if you find them helpful. None of the questions should require extensive calculations. For the questions where motivations are required, you should at a minimum describe the steps that you took to compute the answer. For example, if you did row eliminations, then specify the matrix you start with, and the matrix that you end up with.
- Motivate your work unless a question specifically states that you do not have to.
- Write your answer inside the box given. This is important for GradeScope to be able to correctly scan your exam.

Hints:

- *Question 1:* This question is very close to Question 3 on Section Exam 1. You may find it useful to review the solution to that question. With problems like this one, try to avoid dividing by quantities that could potentially be zero.
- *Question 2:* There was a similar question on Section Exam 2 in 2020. You may find the solutions to that exam of help. Problems (a) through (d) are not intended to be hard.
- *Question 3:* Finding two of the three eigenvalues, and the corresponding eigenspaces is not hard. If you do not easily find the third, you may find the lecture notes for Section 6.3 to be helpful. For this problem, it is probably *not* an effective strategy to attempt computing the characteristic polynomial and then solving the resulting fourth order equation.
- *Question 4:* This problem is a very straight-forward application of material covered in class. It is easy to make arithmetic errors along the way, though, so be sure to carefully check your calculations.
- *Question 5:* Questions (a) and (b) both have short and simple solutions, but you have to be careful with the logic. Theorem 5.10 and the material in Section 5.4 might prove helpful. $\mathcal{P}_n$ refers to the space of all polynomials of degree n or less, as usual. Question (c) is intentionally different from material that we have discussed, so you may find that one more challenging than the others. Note that (c) is worth only 4 points.
- *Question 6:* This is the quirky extra problem with only a small number of points. Part (a) is not difficult, however.

May 7, 2021. Per-Gunnar Martinsson, University of Texas at Austin

Question 1: (18p) Let s and t be real numbers, and consider the linear system

$$\begin{cases} x_1 & +t\,x_2 & +(3+t)\,x_3 & = & 2, \\ -x_1 & -2x_2 & -x_3 & = & s-5, \\ x_1 & +(2t-2)x_2 & +(6+2t)x_3 & = & s, \end{cases} \tag{1}$$

where x_1, x_2, x_3 are the three unknown variables.

(a) For which values of s and t, if any, does the system (1) have a unique solution? Specify the solution set if it exists.

(b) For which values of s and t, if any, does the system (1) not have any solution?

(c) For which values of s and t, if any, does the system (1) have infinitely many solutions? Specify the solution set if it exists.

Question 2: (20p) Let $\mathbf{A}$ be a 3×3 matrix with rows $\{\mathbf{A}_i\}_{i=1}^3$ so that $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}$.

You know that $\det(\mathbf{A}) = 3$. For each matrix given below, specify the value of its determinant in the cases where you have enough information to evaluate it. Please motivate each answer briefly.

$$\det\left(\begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 + 2\mathbf{A}_1 \\ 5\mathbf{A}_3 \end{bmatrix}\right) =$$

$$\det(\mathbf{A} + \mathbf{I}) =$$

$$\det(\mathbf{A} + \mathbf{A}) =$$

$$\det\left(\mathbf{A}^{-1}\right) =$$

$$\det\left(\begin{bmatrix} \mathbf{A}_1 \cdot \mathbf{A}_1 & \mathbf{A}_1 \cdot \mathbf{A}_2 & 2\mathbf{A}_1 \cdot \mathbf{A}_3 \\ \mathbf{A}_2 \cdot \mathbf{A}_1 & \mathbf{A}_2 \cdot \mathbf{A}_2 & 2\mathbf{A}_2 \cdot \mathbf{A}_3 \\ 2\mathbf{A}_3 \cdot \mathbf{A}_1 & 2\mathbf{A}_3 \cdot \mathbf{A}_2 & 4\mathbf{A}_3 \cdot \mathbf{A}_3 \end{bmatrix}\right) =$$

Question 3: (18p) Let $\mathbf{A} = \begin{bmatrix} 0 & -2 & 3 & 3 \\ -2 & 0 & 3 & 3 \\ 3 & 3 & 0 & -2 \\ 3 & 3 & -2 & 0 \end{bmatrix}$, let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and let $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. Observe that $\mathbf{A}$ is symmetric. You know that $\mathbf{A}$ has exactly three distinct eigenvalues, and that the vectors $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of $\mathbf{A}$. Specify all eigenvalues of $\mathbf{A}$, and provide bases for the corresponding eigenspaces.

$\lambda_1 =$	$\lambda_2 =$	$\lambda_3 =$
Basis for $E_{\lambda_1} =$	Basis for $E_{\lambda_2} =$	Basis for $E_{\lambda_3} =$

Motivation:

Question 4a: (10p) Let $\mathbf{x} = [1, -2, 2]$ be a vector, and let $L = \{[t, 2t, t] : t \in \mathbb{R}\}$ be a line in $\mathbb{R}^3$. Compute the distance d between $\mathbf{x}$ and L, where $d = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in L\}$. Explain how you can use the notion of a *projection* of a vector (the function $\mathrm{proj}_{\mathbf{a}}(\mathbf{b})$ that we discussed in Lecture 3) to compute d.

$d =$

Motivation:

Question 4b: (10p) Let $\mathbf{x} = [2, -1, -1]$ be a vector, and let $L = \mathrm{span}\big([1, 1, 1], [1, 2, 1]\big)$ be a plane through the origin in $\mathbb{R}^3$. Determine an orthogonal basis for L, and specify the distance d between $\mathbf{x}$ and L, where $d = \inf\{\|\mathbf{x} - \mathbf{y}\| : \mathbf{y} \in L\}$.

Orthogonal basis for L:

$d =$

Brief motivation:

Question 5: (18p)

(a) (7p) Let $\mathbf{A}$ and $\mathbf{B}$ be a 4×4 matrices. You know that there are two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$. You also know that $\dim(\text{range}(\mathbf{AB})) = 3$. Do you have enough information to determine the *rank* of $\mathbf{A}$? Specify the rank if the answer is yes.

(b) (7p) Let L be a linear map from $\mathcal{P}_2$ to $\mathcal{P}_3$, and let K be a linear map from $\mathcal{P}_3$ to $\mathcal{P}_3$. Define the linear map M from $\mathcal{P}_2$ to $\mathcal{P}_3$ via $M(p) = K(L(p))$. You know that $\ker(L) = \{0\}$ and that $\dim(\text{range}(K)) = 4$. Is M one-to-one? Is M onto?

(c) (4p) Let V denote the linear space consisting of all functions f of the form

$$f(x) = a_0 + \sum_{j=1}^{5} \big(a_j \cos(jx) + b_j \sin(jx)\big),$$

where a_j and b_j are real numbers. Let t denote a real number, and consider the operator L defined by $g = Lf$ where $g(x) = f(x - t)$. For which values of t is L a linear operator from V to V? When is the operator L one-to-one?

Question 6: (6p) Let $\mathbf{A}$ be a matrix of size $m \times k$, and rank k. Set

$$\mathbf{B} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}.$$

In answering this question, you may without proof use that: (i) The matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is invertible. (ii) The matrix $\mathbf{A}$ can be written in the form $\mathbf{A} = \mathbf{UC}$ where $\mathbf{U}$ is an $m \times k$ matrix that satisfies $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}$ and where $\mathbf{C}$ is an invertible matrix. (Both of these facts follow from the statement that $\mathbf{A}$ has rank k.)

(a) Prove that $\mathbf{BA} = \mathbf{I}$.

(b) Prove that $\mathbf{AB}$ is the *orthogonal projection* (as defined in Section 6.2) onto the column space of $\mathbf{A}$.