# Section exam 2 for M341 (52985) Spring 2020

**Released:** Thursday April 16.

**Due:** 5pm, Monday April 20. This is a strict deadline. Please allow yourself a margin!

**Submission logistics:** Submit through GradeScope. Please ensure that you know how this works well before the deadline in case difficulties arise.

**Rules:**

- This is an open book exam.
- The exam should be worked individually. Unlike the homeworks, you are *not* allowed to collaborate.
- You are allowed to use calculators, computers, etc, if you find them helpful. None of the questions should require extensive calculations. For the questions where motivations are required, you should at a minimum describe the steps that you took to compute the answer.
- Motivate your work unless a question specifically states that you do not have to.
- Write your answer inside the box given. This is important for GradeScope to be able to correctly scan your exam.

**Question 1:** (10p) Let $\mathbf{A}$ be a $3 \times 3$ matrix with rows $\{\mathbf{A}_i\}_{i=1}^3$ so that $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}$.

You know that $\det(\mathbf{A}) = 7$. For each matrix given below, specify the value of its determinant in the cases where you have enough information to evaluate it. Please motivate each answer briefly.

$$\det\left(\begin{bmatrix} \mathbf{A}_2 \\ \mathbf{A}_1 \\ \mathbf{A}_3 \end{bmatrix}\right) =$$

$$\det\left(\begin{bmatrix} \mathbf{A}_3 \\ \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}\right) =$$

$$\det\left(\begin{bmatrix} 3\mathbf{A}_2 \\ 3\mathbf{A}_1 \\ 3\mathbf{A}_3 \end{bmatrix}\right) =$$

$$\det\left(\begin{bmatrix} \mathbf{A}_1 + \mathbf{A}_2 \\ \mathbf{A}_1 + \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}\right) =$$

$$\det\left(\begin{bmatrix} \mathbf{A}_1 + \mathbf{A}_2 \\ \mathbf{A}_1 - \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}\right) =$$

**Question 2:** (5p) Let $\mathbf{A}$ be a matrix of size $7 \times 7$, and let $\mathbf{u}$ and $\mathbf{v}$ be vectors of size $7 \times 1$ such that

$$\mathbf{A}\mathbf{u} = \mathbf{u}$$
$$\mathbf{A}\mathbf{v} = 3\mathbf{v}$$
$$\mathbf{A}(\mathbf{u} + \mathbf{v}) = 2(\mathbf{u} + \mathbf{v})$$

Prove that $\mathbf{u}$ and $\mathbf{v}$ must both be the $7 \times 1$ zero vector.

Proof:

**Question 3:** (5p) Compute all eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 5 & 2 \\ -0.5 & -2 & 0.5 \\ 3 & 5 & 0 \end{bmatrix}$.

You may use that for $\mathbf{P} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$, it is the case that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Identity the eigenvalues and the eigenvectors clearly, and briefly motivate how you computed them.

Answer:

**Question 4:** (10p) Let $(\mathbf{v}_i)_{i=1}^3$ denote a basis for $\mathbb{R}^3$. Define vectors $(\mathbf{u}_i)_{i=1}^3$ via

$$\mathbf{u}_1 = \mathbf{v}_1 + \mathbf{v}_2, \qquad \mathbf{u}_2 = \mathbf{v}_2 + \mathbf{v}_3, \qquad \mathbf{u}_3 = \mathbf{v}_3.$$

(a) (4p) Prove that $(\mathbf{u}_i)_{i=1}^3$ is a basis for $\mathbb{R}^3$.

Proof:

(b) (3p) Let $\mathbf{x}$ be a vector in $\mathbb{R}^3$. Since $(\mathbf{u}_i)_{i=1}^3$ and $(\mathbf{v}_i)_{i=1}^3$ are bases for $\mathbb{R}^3$, there exist unique vectors $\mathbf{b} = (b_i)_{i=1}^3$ and $\mathbf{c} = (c_i)_{i=1}^3$ such that

$$\mathbf{x} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3.$$

As we saw in class, there exist $3 \times 3$ matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{b} = \mathbf{P}\mathbf{c}, \qquad \text{and} \qquad \mathbf{c} = \mathbf{Q}\mathbf{b}.$$

Specify the matrices $\mathbf{P}$ and $\mathbf{Q}$. No motivation is required.

$\mathbf{P} =$ 

$\mathbf{Q} =$

(c) (3p) Let $t$ be a real number, and define a set $\mathcal{B} = (\mathbf{w}_i)_{i=1}^3$ through the relations

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \qquad \mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3, \qquad \mathbf{w}_3 = \mathbf{v}_3 + t\mathbf{v}_1.$$

For which values of $t$ is the set $\mathcal{B}$ a basis? Briefly motivate your answer.

Answer:

**Question 5:** (10p) Let $t$ be a real number, and consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ t \\ 1 \\ 2 \end{bmatrix}.$$

Set $L = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$.

(a) (3p) What is the dimension of $L$? Briefly explain how you arrived at the answer. You do not need to provide all details for any calculations.

Answer:

(b) (4p) For which values of $t$ is $\mathbf{w} \in L$? Briefly explain how you arrived at the answer. You do not need to provide all details for any calculations.

Answer:

(c) (3p) Which subsets of the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ form bases for $L$? No motivation is required.

Answer: