Section exam 1 for M341: Linear Algebra and Matrix Theory

8:00am – 9:15am, Feb. 27, 2020. *Closed books. No notes.*

Question 1: (20p) For this question, please write *only the answer*, no motivation. 4p per question.

(a) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right]$

(b) Specify the solution set to the linear system $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$

(c) Let t be a real number and consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & t \\ 3 & 2 \end{bmatrix}$.

For which values of t does $\mathbf{A}$ have an inverse?

(d) Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^7$. State the definition of *the projection of* $\mathbf{x}$ *onto the vector* $\mathbf{y}$.

(e) Circle the statements that are necessarily true.

(i) If the square matrix $\mathbf{A}$ is invertible, then $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.

(ii) Every square matrix can be written as a sum of a symmetric and a skew-symmetric matrix.

(iii) If $\mathbf{x}$ and $\mathbf{y}$ are vectors of the same length, then $\mathbf{x} \cdot \mathbf{y} \geq -\|\mathbf{x}\|\,\|\mathbf{y}\|$.

(iv) Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size. If $\mathbf{AB}$ is invertible, then both $\mathbf{A}$ and $\mathbf{B}$ must be invertible.

Question 2: (10p) Let $\mathbf{a}$ and $\mathbf{b}$ be vectors of the same length. Prove that

$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{a}\|^2.$$

Question 3: (20p) Let $\mathbf{A}$ be a square matrix for which $\mathbf{I}-\mathbf{A}$ is invertible. Prove that for $n = 1, 2, 3, \ldots$ it is the case that

$$\mathbf{A}^0 + \mathbf{A}^1 + \mathbf{A}^2 + \cdots + \mathbf{A}^n = \big(\mathbf{I}-\mathbf{A}\big)^{-1}\big(\mathbf{I}-\mathbf{A}^{n+1}\big).$$

Question 4: (18p) Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ be a 3×2 matrix. Let $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ be 3×3 matrices such that

$$\mathbf{E}_1\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} - 3a_{11} & a_{22} - 3a_{12} \\ a_{31} & a_{32} \end{bmatrix}, \qquad \mathbf{E}_2\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ 5a_{31} & 5a_{32} \end{bmatrix}, \qquad \mathbf{E}_3\mathbf{A} = \begin{bmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

(a) Specify the matrices $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$.

(b) Specify the inverses of the matrices $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$.

No motivations are required.

Hint: $\mathbf{E}_1$ and $\mathbf{E}_2$ are both standard elementary row operations. However, $\mathbf{E}_3$ is not.

Problem 5: (22p) Solve the linear system

$$\begin{array}{rrrcr} x_1 & -x_2 & +x_3 & = & 5 \\ x_1 & -x_2 & +2x_3 & = & 7 \\ -2x_1 & +5x_2 & +4x_3 & = & -7 \end{array}$$

Hint: Remember to verify your solution once you have computed it!

Problem 6: (10p) Specify the inverses of the two matrices

$$\mathbf{A} = \begin{bmatrix} 2 & -8 \\ -2 & 7 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$$

No motivation is required, just give the answers.

Hint: Remember to verify your answers.
