The Gram-Schmidt process

Let $V$ be a subspace of $\mathbb{R}^n$, with a basis $\{w_j\}_{j=1}^k$.

**Gram-Schmidt** is an algorithm for building an orthogonal basis $\{v_j\}_{j=1}^k$ for $V$.

**Step 1:** Set $v_1 = w_1$, and $V_1 = \text{span}(w_1) = \text{span}(v_1)$.

**Step 2:** Set $p_2 = \frac{v_1 \cdot w_2}{\|v_1\|^2} v_1$. \hspace{1cm} \text{\(p_2\) is the projection of \(w_2\) onto \(V_1\).}

Set $v_2 = w_2 - p_2$, and $V_2 = \text{span}(w_1, w_2) = \text{span}(v_1, v_2)$. \hspace{1cm} \text{\(v_2 \in V_1^\perp\) by construction.}

**Step 3:** Set $p_3 = \frac{v_1 \cdot w_3}{\|v_1\|^2} v_1 + \frac{v_2 \cdot w_3}{\|v_2\|^2} v_2$. \hspace{1cm} \text{\(p_3\) is the projection of \(w_3\) onto \(V_2\).}

Set $v_3 = w_3 - p_3$, and $V_3 = \text{span}(w_i)_{i=1}^3 = \text{span}(v_i)_{i=1}^3$. \hspace{1cm} \text{\(v_3 \in V_2^\perp\) by construction.}

**Question:** Could we have “bad luck” and find that $v_3 = 0$?

No, this is not possible. If $v_3 = 0$, then $w_3 = p_3 \in V_2$. Since $V_2$ is spanned by $\{w_1, w_2\}$, this would mean that the set $\{w_1, w_2, w_3\}$ is linearly dependent. This would contract the assumption that the vectors $\{w_i\}$ form a basis.
The Gram-Schmidt process: Let $V$ be a subspace of dimension $k$ in $\mathbb{R}^n$ (e.g. $V = \mathbb{R}^n$).

**Input:** A basis $\{w_j\}_{j=1}^k$ for $V$  

\[ \Rightarrow \]

**Output:** An orthogonal basis $\{v_j\}_{j=1}^k$ for $V$

**Step 1:** $v_1 = w_1$.

**Step 2:** $p_2 = \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1$ 
$v_2 = w_2 - p_2$.

**Step 3:** $p_3 = \frac{w_3 \cdot v_1}{\|v_1\|^2} v_1 + \frac{w_3 \cdot v_2}{\|v_2\|^2} v_2$ 
$v_3 = w_3 - p_3$.

\[
\vdots
\]

**Step $t$:** $p_t = \sum_{i=1}^{t-1} \frac{w_t \cdot v_i}{\|v_i\|^2} v_i$ 
$v_t = w_t - p_t$. 


The Gram-Schmidt process: Let $V$ be a subspace of dimension $k$ in $\mathbb{R}^n$ (e.g. $V = \mathbb{R}^n$).

Input: A basis $\{w_j\}_{j=1}^k$ for $V$  \quad \Rightarrow \quad Output: An orthogonal basis $\{v_j\}_{j=1}^k$ for $V$

Step 1: $v_1 = w_1$.

Step $t$: $p_t = \sum_{i=1}^{t-1} \frac{w_t \cdot v_i}{\|v_i\|^2} v_i$ \quad $v_t = w_t - p_t$.

Properties of the Gram-Schmidt process: The following are true for $t \in \{1, 2, \ldots, k\}$:

- $v_t = w_t + \sum_{j=1}^{t-1} c_j w_j$.
- $\text{span}(v_i)_{i=1}^t = \text{span}(w_i)_{i=1}^t$.
- $p_t$ is the orthogonal projection of $w_t$ onto $\text{span}(w_i)_{i=1}^{t-1}$.
- $v_t$ is orthogonal to $\text{span}(w_i)_{i=1}^{t-1}$.
- $\|v_t\|$ is the distance between $w_t$ and $\text{span}(w_i)_{i=1}^{t-1}$.

It is never zero!
The Gram-Schmidt process with normalization of the vectors:

Let $V$ be a subspace of dimension $k$ in $\mathbb{R}^n$ (e.g. $V = \mathbb{R}^n$).

**Input:** A basis $\{w_j\}_{j=1}^k$ for $V$  

**Output:** An orthonormal basis $\{v_j\}_{j=1}^k$ for $V$

**Step 1:**  
$$v_1 = \frac{w_1}{\|w_1\|}$$

**Step 2:**  
$$p_2 = (w_2 \cdot v_1) v_1$$  
$$v_2 = \frac{w_2 - p_2}{\|w_2 - p_2\|}$$

**Step 3:**  
$$p_3 = (w_3 \cdot v_1) v_1 + (w_3 \cdot v_2) v_2$$  
$$v_3 = \frac{w_3 - p_3}{\|w_3 - p_3\|}$$

...  

**Step $t$:**  
$$p_t = \sum_{i=1}^{t-1} (w_t \cdot v_i) v_i$$  
$$v_t = \frac{w_t - p_t}{\|w_t - p_t\|}$$
Example: Set $w_1 = [1, 2, 2]$, $w_2 = [-1, 0, 2]$, $w_3 = [0, 0, 1]$. Orthogonalize $\{w_i\}_{i=1}^3$!

$v_1 = w_1 = [1, 2, 2]

p_2 = \frac{w_2 \cdot v_1}{\|v_1\|^2} v_1 = \frac{1 \cdot (-1) + 2 \cdot 0 + 2 \cdot 2}{1^2 + 2^2 + 2^2} [1, 2, 2] = \frac{3}{9} [1, 2, 2]

v_2 = w_2 - p_2 = [-1, 0, 2] - \frac{1}{3} [1, 2, 2] = [-4/3, -2/3, 4/3]

p_3 = \frac{w_3 \cdot v_1}{\|v_1\|^2} v_1 + \frac{w_3 \cdot v_2}{\|v_2\|^2} v_2 = \frac{2}{9} [1, 2, 2] + \frac{4/3}{4} [-4/3, -2/3, 4/3]

v_3 = w_3 - p_3 = [2/9, -2/9, 1/9]