

Let A be an $m \times n$ matrix, and let \vec{B} be an $n \times 1$ vector.

We seek to solve $A\vec{x} = \vec{B}$.

Step 1: Form the extended coefficient matrix $[A | \vec{B}]$

Step 2: Start to form the RREF using row operations.

If you find an inconsistent row, then stop. \rightarrow NO SOLUTIONS.

If not, then you will end up with the RREF.

You can now classify the system:

{ Unique solⁿ
or
Ininitely many solⁿs

If you want to compute the solⁿs explicitly, then continue
to compute the RREF. (Or just do back substitution.)

Defⁿ Let A be an $m \times n$ matrix with RREF \tilde{A} .

The rank of A is the number of pivots in \tilde{A} .

Note: The rank is the number of nonzero rows.

This is the "true" number of equations.

Let $[A | B]$ be a linear system with RREF $[\tilde{A} | \tilde{B}]$.

The system is INCONSISTENT if $\text{rank } \tilde{A} < \text{rank } [\tilde{A} | \tilde{B}]$.
(That is, if there is a row with the pivot to the right of the line.)

If the system is CONSISTENT, then:

There is a unique solⁿ $\Leftrightarrow \text{rank}(A) = n$

(That is, there are no "free variables".)

SPECIAL CASE

Consider the linear system

A "homogeneous" \rightarrow
linear system

$$A\vec{x} = \vec{0}$$

\uparrow \downarrow
 $m \times n$ $n \times 1$ vector

The system is ALWAYS consistent (since $\vec{0}$ is a solⁿ).

In this case, the only "classification question" is whether the solⁿ is unique. We have:

Unique solⁿ \Leftrightarrow no free variables $\Leftrightarrow \text{rank}(A) = n$

HOMOGENEOUS & PARTICULAR SOLUTIONS

Consider a linear system $A\vec{x} = \vec{B}$. (*)

Let \vec{x}_p be one soln to (*). That is, $A\vec{x}_p = \vec{B}$.

Then: (a) If \vec{x}_h satisfies $A\vec{x}_h = 0$, then

$\vec{x} = \vec{x}_p + \vec{x}_h$ also solves (*)

(b) If \vec{x} is a soln of (*), then

\vec{x} is of the form $\vec{x} = \vec{x}_h + \vec{x}_p$ for
some \vec{x}_h s.t. $A\vec{x}_h = 0$.

SECTION 3.3 Recall that there are three basic row operations:

- (i) Multiplying a row by a nonzero scalar c . $\langle i \rangle \leftarrow c \langle i \rangle$
- (ii) Adding a scalar multiple of one row to another. $\langle i \rangle \leftarrow \langle i \rangle + c \langle j \rangle$
- (iii) Swapping two rows. $\langle i \rangle \leftrightarrow \langle j \rangle$.

Important Fact: Every row operation is reversible!

Inverse of (i): $\langle i \rangle \leftarrow \frac{1}{c} \langle i \rangle$

Inverse of (ii): $\langle i \rangle \leftarrow \langle i \rangle - c \langle j \rangle$

Inverse of (iii): $\langle j \rangle \leftrightarrow \langle i \rangle$ (The same as $\langle i \rangle \leftrightarrow \langle j \rangle$!)

Every inverse is also a row operation. (Of the same type, in fact.)