Question 1: (10p) Consider the matrices

\[
A = \begin{bmatrix}
1 & 0 & 3 & -1 \\
1 & 1 & 0 & 0 \\
-2 & -1 & -4 & 2
\end{bmatrix} \quad B = \begin{bmatrix}
1 & 0 & 3 & -1 \\
0 & 1 & -2 & 1 \\
0 & -1 & 2 & 0
\end{bmatrix} \quad C = \begin{bmatrix}
1 & 0 & 3 & -1 \\
0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad D = \begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

There exist 3 \times 3 matrices \(R_1, R_2\), and \(R_3\) such that
\[
B = R_1 A, \quad C = R_2 B, \quad D = R_3 C.
\]

Each matrix \(R_i\) is a product of at most two elementary row operations.

(a) (2p) Specify the matrices \(R_1, R_2,\) and \(R_3\). No motivation required.

(b) (2p) Set \(R = R_3 R_2 R_1\), so that \(D = RA\). Specify \(R^{-1}\). No motivation required.

(c) (2p) Set \(Y = \begin{bmatrix} 3 \\ 2 \\ -8 \end{bmatrix}\). Specify the solution set to the equation \(AX = Y\). No motivation required.

(d) (2p) Specify a basis for the kernel of \(A\). No motivation required.

(e) (2p) Suppose that \(S\) is a matrix such that \(D = SA\). Is it necessarily the case that \(S = R\), where \(R\) is the matrix defined in (b)? Motivate briefly.

Solution:

(a) \(R_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\) \(R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\) \(R_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\)

(b) \(R = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\) \(R^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix}\)

(c) The system \(AX = Y\) is equivalent to the system \(RAX = RY\), which is \(DX = RY\).

We have \(RY = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}\), so the system we need to solve is
\[
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}.
\]

The free variable is \(x_3\). For clarity, set \(t = x_3\), and then we read off the solution
\[
X = \begin{bmatrix} -3t \\ 2 + 2t \\ t \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} t.
\]

(d) From problem (c), we find that \(\ker(A) = \text{span}([-3, 2, 1, 0])\).

(e) Yes, it must be the case that \(R = S\). To prove this, set \(T = R - S\). Then subtracting the equations \(RA = D\) and \(SA = D\), we get the equation \(TA = 0\). The rows of \(A\) are linearly independent (since the rank of \(A\) is 3), so the only solution is \(T = 0\).

Note: If the matrix \(A\) does not have a pivot element in every row (so that \(\text{rank}(A) = 3\) in this case), then the matrix \(R\) is not necessarily unique.
**Question 2:** (10p) Consider the matrix $A = \begin{pmatrix} -7 & 0 & 4 \\ -8 & 1 & 4 \\ -8 & 0 & 5 \end{pmatrix}$.

(a) (4p) Specify the characteristic polynomial of $A$ and specify its eigenvalues. No motivation required.

(b) (3p) Specify a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$. Briefly describe how you computed them.

(c) (3p) Let $p$ be a positive integer. Compute $A^p$. Your answer should be a single $3 \times 3$ matrix whose entries depend on $p$. (In other words, please do not give an answer in the form of a product of any matrices.) Describe how you arrived at your answer.

**Solution:**

(a) We compute and factorize the characteristic polynomial:

$$p_A(\lambda) = \det \begin{vmatrix} \lambda + 7 & 0 & -4 \\ 8 & \lambda - 1 & -4 \\ 8 & 0 & \lambda - 5 \end{vmatrix} = (\lambda + 7)(\lambda - 1)(\lambda - 5) + 4(\lambda - 1)8$$

$$= (\lambda - 1)(\lambda^2 + 2\lambda - 35) + (\lambda - 1)32 = (\lambda - 1)(\lambda^2 + 2\lambda - 3) = (\lambda - 1)^2(\lambda + 3).$$

We see that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -3$.

(b) To find the eigenvectors, we solve the linear system $(\lambda I - A)x = 0$.

For $\lambda_1 = 1$, we get $\begin{pmatrix} 8 & 0 & -4 \\ 8 & 0 & -4 \\ 8 & 0 & -4 \end{pmatrix}$. Two linearly independent solutions; e.g. $u_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -3$, we get $\begin{pmatrix} 4 & 0 & -4 \\ 8 & -4 & -4 \\ 8 & 0 & -8 \end{pmatrix}$. One solution is $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) Using our results in (b), we know that $A = VD V^{-1}$ where

$$V = [u_1 \ u_2 \ v] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$ 

It follows that

$$A^p = V D^p V^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-3)^p \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 + 2(-3)^p & 0 & 1 - (-3)^p \\ -2 + 2(-3)^p & 1 & 1 - (-3)^p \\ -2 + 2(-3)^p & 0 & 2 - (-3)^p \end{bmatrix}.$$
Question 3: Let \( \mathbf{x} = [x_1, x_2, x_3] \) be a given nonzero vector in \( \mathbb{R}^3 \). Define the matrix

\[
\mathbf{A}_x = \begin{bmatrix}
0 & -x_3 & x_2 \\
-x_3 & 0 & -x_1 \\
x_2 & x_1 & 0
\end{bmatrix}.
\]

In answering this question, you may assume that \( x_3 \neq 0 \). No motivation is required for (a) – (d).

(a) (2p) Specify the characteristic polynomial of \( \mathbf{A}_x \). (Observe that this will be a polynomial whose coefficients depend on the given vector \( \mathbf{x} \).) Specify the real eigenvalues of \( \mathbf{A}_x \).

(b) (2p) Specify the kernel of \( \mathbf{A}_x \).

(c) (1p) Specify the rank of \( \mathbf{A}_x \).

(d) (2p) Given a vector \( \mathbf{y} \in \mathbb{R}^3 \), set \( \mathbf{z} = \mathbf{A}_x \mathbf{y} \). Specify \( \mathbf{z} \cdot \mathbf{x} \) and \( \mathbf{z} \cdot \mathbf{y} \).

(e) (3p) Specify the range of \( \mathbf{A}_x \). Prove that your answer is correct.

Solution:

(a) \( p_\mathbf{A}(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \det \begin{bmatrix}
\lambda & x_3 & -x_2 \\
-x_3 & \lambda & x_1 \\
x_2 & -x_1 & \lambda
\end{bmatrix} = \lambda^3 + \lambda x_1^2 + \lambda x_2^2 + \lambda x_3^2 = \lambda^2 + \|\mathbf{x}\|^2 \)

The only real solution of \( \lambda(\lambda^2 + \|\mathbf{x}\|^2) = 0 \) is: \( \lambda = 0 \)

(b) We solve \( \mathbf{A}_x \mathbf{y} = \mathbf{0} \), which takes the form

\[-x_3 y_2 + x_2 y_3 = 0 \\
x_3 y_1 - x_1 y_3 = 0 \\
-x_2 y_1 + x_1 y_2 = 0\]

The first two equations say that \( y_2 = (x_2/x_3)y_3 \) and \( y_1 = (x_1/x_3)y_3 \) so the general solution is

\[
\mathbf{y} = \begin{bmatrix}
(x_1/x_3)y_3 \\
(x_2/x_3)y_3 \\
y_3
\end{bmatrix} = \frac{y_3}{x_3} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \frac{y_3}{x_3} \mathbf{x}.
\]

We see that \( \mathbf{A}_x \mathbf{y} = \mathbf{0} \) iff \( \mathbf{y} \) is a multiple of \( \mathbf{x} \), so: \( \ker(\mathbf{A}_x) = \text{span}(\mathbf{x}) \)

(c) Since the kernel has dimension 1, we find that \( \text{rank} = 3 - 1 = 2 \)

(d) This is a direct calculation:

\[
\mathbf{A}_x \mathbf{y} \cdot \mathbf{x} = \begin{bmatrix}
-x_3 y_2 + x_2 y_3 \\
x_3 y_1 - x_1 y_3 \\
-x_2 y_1 + x_1 y_2
\end{bmatrix} \cdot \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = x_1(-x_3 y_2 + x_2 y_3) + x_2(x_3 y_1 - x_1 y_3) + x_3(-x_2 y_1 + x_1 y_2) = 0
\]

\[
\mathbf{A}_x \mathbf{y} \cdot \mathbf{x} = \begin{bmatrix}
-x_3 y_2 + x_2 y_3 \\
x_3 y_1 - x_1 y_3 \\
-x_2 y_1 + x_1 y_2
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y_1 \\
y_2 \\
y_3
\end{bmatrix} = y_1(-x_3 y_2 + x_2 y_3) + y_2(x_3 y_1 - x_1 y_3) + y_3(-x_2 y_1 + x_1 y_2) = 0
\]
(e) \( \text{ran}(A_x) = (\text{span}(x))^\perp \)

Observe that a vector \( b \) belongs to the span of \( A_x \) iff the equation \( A_x y = b \) has a solution. We find

\[
\begin{bmatrix}
0 & -x_3 & x_2 & b_1 \\
x_3 & 0 & -x_1 & b_2 \\
-x_2 & x_1 & 0 & b_3 \\
\end{bmatrix}
\sim
\begin{bmatrix}
x_3 & 0 & -x_1 & b_2 \\
0 & -x_3 & x_2 & b_1 \\
-x_2 & x_1 & 0 & b_3 \\
\end{bmatrix}
\sim
\begin{bmatrix}
x_3 & 0 & -x_1 & b_2 \\
0 & -x_3 & x_2 & b_1 \\
0 & x_1 & -x_1 x_2 / x_3 & b_3 + x_2 b_x / x_3 \\
\end{bmatrix}
\sim
\begin{bmatrix}
x_3 & 0 & -x_1 & b_2 \\
0 & -x_3 & x_2 & b_1 \\
0 & 0 & 0 & b_3 + x_2 b_x / x_3 + x_1 b_1 / x_3 \\
\end{bmatrix}.
\]

We see that the system is consistent if and only if

\[ b_3 + x_2 b_2 / x_3 + x_1 b_1 / x_3 = 0. \]

Multiplying by \( x_3 \), we see that the system is consistent if and only if \( b \cdot x = 0. \)

**Alternative solution to (e):**

\[
\text{ran}(A_x) = \text{span} \left\{ \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \begin{bmatrix} -x_3 \\ 0 \\ -x_1 \end{bmatrix} \right\}
\]

The motivation here is that if you row reduce \( A \) under the assumption that \( x_3 \neq 0 \), then you find that columns 1 and 2 are the pivot columns, and then columns 1 and 2 of \( A \) form a basis for the range.
**Question 4:** (10p) Let $\mathcal{P}_r$ denote the vector space of all polynomials of degree at most $r$, as usual. Consider the following five maps between polynomial spaces:

- $T_1$ maps $\mathcal{P}_3$ to $\mathcal{P}_2$, according to the formula $[T_1p](x) = p'(x)$.
- $T_2$ maps $\mathcal{P}_3$ to $\mathcal{P}_3$, according to the formula $[T_2p](x) = xp'(x) + 1$.
- $T_3$ maps $\mathcal{P}_2$ to $\mathcal{P}_4$, according to the formula $[T_3p](x) = x^2 p(x) + \int_0^x p(y) \, dy$.
- $T_4$ maps $\mathcal{P}_3$ to $\mathcal{P}_2$, according to the formula $[T_4p](x) = \frac{p(x) - p(0)}{x}$.
- $T_5$ maps $\mathcal{P}_3$ to $\mathcal{P}_4$, according to the formula $[T_5p](x) = \left(\frac{p(x) - p(0)}{x}\right)^2$.

(a) (5p) For each map $T_j$ check the box that best describes the map.
*Note: There should be ONLY ONE check in each column.*

<table>
<thead>
<tr>
<th>$T_j$ is not linear:</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$T_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_j$ is linear and onto but not one-to-one:</td>
<td>X</td>
<td></td>
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<tr>
<td>$T_j$ is linear and one-to-one but not onto:</td>
<td></td>
<td>X</td>
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<tr>
<td>$T_j$ is linear and neither one-to-one nor onto:</td>
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<tr>
<td>$T_j$ is linear and both one-to-one and onto:</td>
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</tbody>
</table>

(b) (5p) Let $\mathcal{B}_r = (1, x, x^2, \ldots, x^r)$ denote an ordered bases for $\mathcal{P}_r$. For the maps that were linear, specify the matrix of the map with respect to $\mathcal{B}_r$. (For the maps that are not linear, leave blank!)

The matrix for $T_1$ is:

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix}.
$$

The matrix for $T_3$ is:

$$
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 1/2 & 0 \\
0 & 1 & 1/3 \\
0 & 0 & 1
\end{bmatrix}.
$$

The matrix for $T_4$ is:

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

*Note: Observe that $T_20 \neq 0$ and $T_5(2p) = 4T_5(p)$, so these two cannot be linear.*
Question 5: (10p) Consider the three vectors 
\[ a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}. \]

(a) (5p) Use the Gram-Schmidt process to compute an orthonormal set \( \{q_1, q_2, q_3\} \) such that
- \( \text{span}(q_1) = \text{span}(a_1) \)
- \( \text{span}(q_1, q_2) = \text{span}(a_1, a_2) \)
- \( \text{span}(q_1, q_2, q_3) = \text{span}(a_1, a_2, a_3) \)

No motivation is required for (a) and (b).

(b) (3p) Set 
\[ A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}. \]

Define 
\[ R = Q^T A, \]
so that 
\[ A = QR. \]
Specify \( R \).

(c) (2p) Let \( m \) and \( n \) denote integers such that \( 0 < n \leq m \). Let \( \{a_j\}_{j=1}^n \) denote a set of linearly independent vectors in \( \mathbb{R}^m \), and let \( \{q_j\}_{j=1}^n \) denote an orthonormal set that results from applying the Gram-Schmidt process to the vectors \( \{a_j\}_{j=1}^n \). Let \( A \) and \( Q \) denote the \( m \times n \) matrices whose columns are the vectors \( \{a_j\}_{j=1}^n \) and \( \{q_j\}_{j=1}^n \), respectively. Set \( R = Q^T A \). Prove that \( R \) is upper triangular.

Solution:

(a) \[ q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad q_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}. \]

(b) 
\[ R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \sqrt{3} & 0 \\ 0 & \sqrt{2} & 2\sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix}. \]

(c) Consider the entry \( R(i, j) \) for \( i > j \). We need to prove that \( R(i, j) = 0 \). Observe that 
\[ R(i, j) = q_i^T a_j. \]
In other words, \( R(i, j) \) is the inner product between \( q_i \) and \( a_j \). By construction, \( q_i \) is orthogonal to \( \text{span}(a_1, a_2, \ldots, a_{i-1}) \). In particular, \( q_i \) is orthogonal to \( a_j \) since \( j \leq i - 1 \).