Final exam for M341 (52985) Spring 2020 — Solutions

Question 1: (10p) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & 1 & 1 & 0 \\ -2 & -1 & -4 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There exist 3×3 matrices $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ such that

$$\mathbf{B} = \mathbf{R}_1\mathbf{A}, \qquad \mathbf{C} = \mathbf{R}_2\mathbf{B}, \qquad \mathbf{D} = \mathbf{R}_3\mathbf{C}.$$

Each matrix $\mathbf{R}_i$ is a product of at most two elementary row operations.

(a) (2p) Specify the matrices $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$. No motivation required.

(b) (2p) Set $\mathbf{R} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$, so that $\mathbf{D} = \mathbf{R}\mathbf{A}$. Specify $\mathbf{R}^{-1}$. No motivation required.

(c) (2p) Set $\mathbf{Y} = \begin{bmatrix} 3 \\ 2 \\ -8 \end{bmatrix}$. Specify the solution set to the equation $\mathbf{AX} = \mathbf{Y}$. No motivation required.

(d) (2p) Specify a basis for the kernel of $\mathbf{A}$. No motivation required.

(e) (2p) Suppose that $\mathbf{S}$ is a matrix such that $\mathbf{D} = \mathbf{SA}$. Is it necessarily the case that $\mathbf{S} = \mathbf{R}$, where $\mathbf{R}$ is the matrix defined in (b)? Motivate briefly.

Solution:

(a)
$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad \mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{R}_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b)
$$\mathbf{R} = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{R}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$$

(c) The system $\mathbf{AX} = \mathbf{Y}$ is equivalent to the system $\mathbf{RAX} = \mathbf{RY}$, which is $\mathbf{DX} = \mathbf{RY}$.

We have $\mathbf{RY} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$, so the system we need to solve is

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right].$$

The free variable is x_3. For clarity, set $t = x_3$, and then we read off the solution

$$\mathbf{X} = \begin{bmatrix} -3t \\ 2+2t \\ t \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} t.$$

(d) From problem (c), we find that $\ker(\mathbf{A}) = \text{span}([-3, 2, 1, 0])$.

(e) Yes, it must be the case that $\mathbf{R} = \mathbf{S}$. To prove this, set $\mathbf{T} = \mathbf{R} - \mathbf{S}$. Then substracting the equations $\mathbf{RA} = \mathbf{D}$ and $\mathbf{SA} = \mathbf{D}$, we get the equation $\mathbf{TA} = \mathbf{0}$. The rows of $\mathbf{A}$ are linearly independent (since the rank of $\mathbf{A}$ is 3), so the only solution is $\mathbf{T} = \mathbf{0}$.

*Note: If the matrix $\mathbf{A}$ does not have a pivot element in every row (so that rank($\mathbf{A}$) = 3 in this case), then the matrix $\mathbf{R}$ is **not** necessarily unique.*

Question 2: (10p) Consider the matrix $\mathbf{A} = \begin{bmatrix} -7 & 0 & 4 \\ -8 & 1 & 4 \\ -8 & 0 & 5 \end{bmatrix}$.

(a) (4p) Specify the characteristic polynomial of $\mathbf{A}$ and specify its eigenvalues. No motivation required.

(b) (3p) Specify a basis for $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{A}$. Briefly describe how you computed them.

(c) (3p) Let p be a positive integer. Compute $\mathbf{A}^p$. Your answer should be a single 3×3 matrix whose entries depend on p. (In other words, please do not give an answer in the form of a product of any matrices.) Describe how you arrived at your answer.

Solution:

(a) We compute and factorize the characteristic polynomial:

$$p_{\mathbf{A}}(\lambda) = \det \begin{bmatrix} \lambda+7 & 0 & -4 \\ 8 & \lambda-1 & -4 \\ 8 & 0 & \lambda-5 \end{bmatrix} = (\lambda+7)(\lambda-1)(\lambda-5) + 4(\lambda-1)8$$
$$= (\lambda-1)(\lambda^2+2\lambda-35) + (\lambda-1)32 = (\lambda-1)\big(\lambda^2+2\lambda-3\big) = (\lambda-1)^2(\lambda+3).$$

We see that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -3$.

(b) To find the eigenvectors, we solve the linear system $\big(\lambda\mathbf{I} - \mathbf{A}\big)\mathbf{x} = \mathbf{0}$.

For $\lambda_1 = 1$, we get $\left[\begin{array}{ccc|c} 8 & 0 & -4 & 0 \\ 8 & 0 & -4 & 0 \\ 8 & 0 & -4 & 0 \end{array}\right]$. Two linearly independent solutions; e.g. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

For $\lambda_2 = -3$, we get $\left[\begin{array}{ccc|c} 4 & 0 & -4 & 0 \\ 8 & -4 & -4 & 0 \\ 8 & 0 & -8 & 0 \end{array}\right]$. One solution is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) Using our results in (b), we know that

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

where

$$\mathbf{V} = \big[\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{v}\big] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

It follows that

$$\mathbf{A}^p = \mathbf{V}\mathbf{D}^p\mathbf{V}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-3)^p \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1+2(-3)^p & 0 & 1-(-3)^p \\ -2+2(-3)^p & 1 & 1-(-3)^p \\ -2+2(-3)^p & 0 & 2-(-3)^p \end{bmatrix}.$$

Question 3: Let $\mathbf{x} = [x_1,\, x_2,\, x_3]$ be a given nonzero vector in $\mathbb{R}^3$. Define the matrix

$$\mathbf{A}_{\mathbf{x}} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

In answering this question, you may assume that $x_3 \neq 0$. No motivation is required for (a) – (d).

(a) (2p) Specify the characteristic polynomial of $\mathbf{A}_{\mathbf{x}}$. (Observe that this will be a polynomial whose coefficients depend on the given vector $\mathbf{x}$.) Specify the real eigenvalues of $\mathbf{A}_{\mathbf{x}}$.

(b) (2p) Specify the kernel of $\mathbf{A}_{\mathbf{x}}$.

(c) (1p) Specify the rank of $\mathbf{A}_{\mathbf{x}}$.

(d) (2p) Given a vector $\mathbf{y} \in \mathbb{R}^3$, set $\mathbf{z} = \mathbf{A}_{\mathbf{x}}\,\mathbf{y}$. Specify $\mathbf{z} \cdot \mathbf{x}$ and $\mathbf{z} \cdot \mathbf{y}$.

(e) (3p) Specify the range of $\mathbf{A}_{\mathbf{x}}$. Prove that your answer is correct.

Solution:

(a) $p_{\mathbf{A}}(\lambda) = \det\big(\lambda\mathbf{I} - \mathbf{A}\big) = \det\begin{bmatrix} \lambda & x_3 & -x_2 \\ -x_3 & \lambda & x_1 \\ x_2 & -x_1 & \lambda \end{bmatrix} = \lambda^3 + \lambda x_1^2 + \lambda x_2^2 + \lambda x_3^2 = \lambda\big(\lambda^2 + \|\mathbf{x}\|^2\big)$

The only real solution of $\lambda\big(\lambda^2 + \|\mathbf{x}\|^2\big) = 0$ is: $\boxed{\lambda = 0}$

(b) We solve $\mathbf{A}_{\mathbf{x}}\mathbf{y} = \mathbf{0}$, which takes the form

$$\begin{aligned} -x_3y_2 + x_2y_3 &= 0 \\ x_3y_1 - x_1y_3 &= 0 \\ -x_2y_1 + x_1y_2 &= 0 \end{aligned}$$

The first two equations say that $y_2 = (x_2/x_3)y_3$ and $y_1 = (x_1/x_3)y_3$ so the general solution is

$$\mathbf{y} = \begin{bmatrix} (x_1/x_3)y_3 \\ (x_2/x_3)y_3 \\ y_3 \end{bmatrix} = \frac{y_3}{x_3}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{y_3}{x_3}\mathbf{x}.$$

We see that $\mathbf{A}_{\mathbf{x}}\mathbf{y} = \mathbf{0}$ iff $\mathbf{y}$ is a multiple of $\mathbf{x}$, so: $\boxed{\ker(\mathbf{A}_{\mathbf{x}}) = \operatorname{span}(\mathbf{x})}$

(c) Since the kernel has dimension 1, we find that $\boxed{\text{rank} = 3 - 1 = 2}$.

(d) This is a direct calculation:

$$\mathbf{A}_{\mathbf{x}}\mathbf{y} \cdot \mathbf{x} = \begin{bmatrix} -x_3y_2 + x_2y_3 \\ x_3y_1 - x_1y_3 \\ -x_2y_1 + x_1y_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1(-x_3y_2 + x_2y_3) + x_2(x_3y_1 - x_1y_3) + x_3(-x_2y_1 + x_1y_2) = 0$$

$$\mathbf{A}_{\mathbf{x}}\mathbf{y} \cdot \mathbf{x} = \begin{bmatrix} -x_3y_2 + x_2y_3 \\ x_3y_1 - x_1y_3 \\ -x_2y_1 + x_1y_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1(-x_3y_2 + x_2y_3) + y_2(x_3y_1 - x_1y_3) + y_3(-x_2y_1 + x_1y_2) = 0$$

(e) $\boxed{\operatorname{ran}(\mathbf{A_x}) = \left(\operatorname{span}(\mathbf{x})\right)^\perp}$

Observe that a vector **b** belongs to the span of $\mathbf{A_x}$ iff the equation $\mathbf{A_x y} = \mathbf{b}$ has a solution. We find

$$\left[\begin{array}{rrr|l} 0 & -x_3 & x_2 & b_1 \\ x_3 & 0 & -x_1 & b_2 \\ -x_2 & x_1 & 0 & b_3 \end{array}\right] \sim \left[\begin{array}{rrr|l} x_3 & 0 & -x_1 & b_2 \\ 0 & -x_3 & x_2 & b_1 \\ -x_2 & x_1 & 0 & b_3 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|r} x_3 & 0 & -x_1 & b_2 \\ 0 & -x_3 & x_2 & b_1 \\ 0 & x_1 & -x_1x_2/x_3 & b_3 + x_2b_x/x_3 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|r} x_3 & 0 & -x_1 & b_2 \\ 0 & -x_3 & x_2 & b_1 \\ 0 & 0 & 0 & b_3 + x_2b_2/x_3 + x_1b_1/x_3 \end{array}\right].$$

We see that the system is consistent if and only if

$$b_3 + x_2b_2/x_3 + x_1b_1/x_3 = 0.$$

Multiplying by x_3, we see that the system is consistent if and only if $\mathbf{b} \cdot \mathbf{x} = 0$.

Alternative solution to (e): $\boxed{\operatorname{ran}(\mathbf{A_x}) = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix}, \begin{bmatrix} -x_3 \\ 0 \\ -x_1 \end{bmatrix} \right\}}$

The motivation here is that if you row reduce **A** under the assumption that $x_3 \neq 0$, then you find that columns 1 and 2 are the pivot columns, and then columns 1 and 2 of **A** form a basis for the range.

Question 4: (10p) Let $\mathcal{P}_r$ denote the vector space of all polynomials of degree at most r, as usual. Consider the following five maps between polynomial spaces:

- T_1 maps $\mathcal{P}_3$ to $\mathcal{P}_2$, according to the formula $[T_1 p](x) = p'(x)$.
- T_2 maps $\mathcal{P}_3$ to $\mathcal{P}_3$, according to the formula $[T_2 p](x) = x\,p'(x) + 1$.
- T_3 maps $\mathcal{P}_2$ to $\mathcal{P}_4$, according to the formula $[T_3 p](x) = x^2\,p(x) + \int_0^x p(y)\,dy$.
- T_4 maps $\mathcal{P}_3$ to $\mathcal{P}_2$, according to the formula $[T_4 p](x) = \dfrac{p(x) - p(0)}{x}$.
- T_5 maps $\mathcal{P}_3$ to $\mathcal{P}_4$, according to the formula $[T_5 p](x) = \left(\dfrac{p(x) - p(0)}{x}\right)^2$.

(a) (5p)For each map T_j check the box that best describes the map.
Note: There should be ONLY ONE check in each column.

	T_1	T_2	T_3	T_4	T_5
T_j is **not** linear:		X			X
T_j is linear and onto but not one-to-one:	X			X	
T_j is linear and one-to-one but not onto:			X		
T_j is linear and neither one-to-one nor onto:					
T_j is linear and both one-to-one and onto:					

(b) (5p) Let $\mathcal{B}_r = (1,\, x,\, x^2,\, \dots,\, x^r)$ denote an ordered bases for $\mathcal{P}_r$. For the maps that were linear, specify the matrix of the map with respect to $\mathcal{B}_r$. (For the maps that are not linear, leave blank!)

The matrix for T_1 is: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

The matrix for T_3 is: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix}$.

The matrix for T_4 is: $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Note: Observe that $T_2 0 \neq 0$ and $T_5(2p) = 4T_5(p)$, so these two cannot be linear.

Question 5: (10p) Consider the three vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$.

(a) (5p) Use the Gram-Schmidt process to compute an ortho**normal** set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ such that

- $\text{span}(\mathbf{q}_1) = \text{span}(\mathbf{a}_1)$.
- $\text{span}(\mathbf{q}_1, \mathbf{q}_2) = \text{span}(\mathbf{a}_1, \mathbf{a}_2)$.
- $\text{span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

No motivation is required for (a) and (b).

(b) (3p) Set $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 \, \mathbf{a}_2 \, \mathbf{a}_3\end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix}\mathbf{q}_1 \, \mathbf{q}_2 \, \mathbf{q}_3\end{bmatrix}$. Define $\mathbf{R} = \mathbf{Q}^{\mathrm{T}}\mathbf{A}$, so that $\mathbf{A} = \mathbf{QR}$. Specify $\mathbf{R}$.

(c) (2p) Let m and n denote integers such that $0 < n \leq m$. Let $\{\mathbf{a}_j\}_{j=1}^n$ denote a set of linearly independent vectors in $\mathbb{R}^m$, and let $\{\mathbf{q}_j\}_{j=1}^n$ denote an orthonormal set that results from applying the Gram-Schmidt process to the vectors $\{\mathbf{a}_j\}_{j=1}^n$. Let $\mathbf{A}$ and $\mathbf{Q}$ denote the $m \times n$ matrices whose columns are the vectors $\{\mathbf{a}_j\}_{j=1}^n$ and $\{\mathbf{q}_j\}_{j=1}^n$, respectively. Set $\mathbf{R} = \mathbf{Q}^{\mathrm{T}}\mathbf{A}$. Prove that $\mathbf{R}$ is upper triangular.

Solution:

(a) $\mathbf{q}_1 = \dfrac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\qquad \mathbf{q}_2 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\qquad \mathbf{q}_3 = \dfrac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

(b)

$$\mathbf{R} = \mathbf{Q}^{\mathrm{T}}\mathbf{A} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & \sqrt{3} & 0 \\ 0 & \sqrt{2} & 2\sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix}$$

(c) Consider the entry $\mathbf{R}(i, j)$ for $i > j$. We need to prove that $\mathbf{R}(i, j) = 0$. Observe that

$$\mathbf{R}(i, j) = \mathbf{q}_i^{\mathrm{T}}\mathbf{a}_j.$$

In other words, $\mathbf{R}(i, j)$ is the inner product between $\mathbf{q}_i$ and $\mathbf{a}_j$. By construction, $\mathbf{q}_i$ is orthogonal to $\text{span}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{i-1})$. In particular, $\mathbf{q}_i$ is orthogonal to $\mathbf{a}_j$ since $j \leq i - 1$.