Final exam for M341 (52985) Spring 2020

Released: Thursday May 14.

Due: Noon, Monday May 18. This is a strict deadline. Please allow yourself a margin!

Submission logistics: Submit through GradeScope. Please ensure that you know how this works well before the deadline in case difficulties arise.

Rules:

- This is an open book exam.
- The exam should be worked individually. Unlike the homeworks, you are *not* allowed to collaborate.
- You are allowed to use calculators, computers, etc, if you find them helpful. None of the questions should require extensive calculations. For the questions where motivations are required, you should at a minimum describe the steps that you took to compute the answer. For example, if you did row eliminations, then specify the matrix you start with, and the matrix that you end up with.
- Motivate your work unless a question specifically states that you do not have to.
- Write your answer inside the box given. This is important for GradeScope to be able to correctly scan your exam.

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Question 1: (10p) Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 1 & 1 & 1 & 0 \\ -2 & -1 & -4 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

There exist 3×3 matrices $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ such that

$$\mathbf{B} = \mathbf{R}_1\mathbf{A}, \qquad \mathbf{C} = \mathbf{R}_2\mathbf{B}, \qquad \mathbf{D} = \mathbf{R}_3\mathbf{C}.$$

Each matrix $\mathbf{R}_i$ is a product of at most two elementary row operations.

(a) (2p) Specify the matrices $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$. No motivation required.

$\mathbf{R}_1 =$ $\qquad$ $\mathbf{R}_2 =$ $\qquad$ $\mathbf{R}_3 =$

(b) (2p) Set $\mathbf{R} = \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1$, so that $\mathbf{D} = \mathbf{RA}$. Specify $\mathbf{R}^{-1}$. No motivation required.

$\mathbf{R}^{-1} =$

(c) (2p) Set $\mathbf{Y} = \begin{bmatrix} 3 \\ 2 \\ -8 \end{bmatrix}$. Specify the solution set to the equation $\mathbf{AX} = \mathbf{Y}$. No motivation required.

The solution set is:

(d) (2p) Specify a basis for the kernel of $\mathbf{A}$. No motivation required.

Basis for $\ker(\mathbf{A}) =$

(e) (2p) Suppose that $\mathbf{S}$ is a matrix such that $\mathbf{D} = \mathbf{SA}$. Is it necessarily the case that $\mathbf{S} = \mathbf{R}$, where $\mathbf{R}$ is the matrix defined in (b)? Motivate briefly.

Answer (yes/no):

Motivation:

Question 2: (10p) Consider the matrix $\mathbf{A} = \begin{bmatrix} -7 & 0 & 4 \\ -8 & 1 & 4 \\ -8 & 0 & 5 \end{bmatrix}$.

(a) (4p) Specify the characteristic polynomial of $\mathbf{A}$ and specify its eigenvalues. No motivation required.

Characteristic polynomial of $\mathbf{A}$ =

Eigenvalues of $\mathbf{A}$ =

(b) (3p) Specify a basis for $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{A}$. Briefly describe how you computed them.

Eigenvectors of $\mathbf{A}$ =

How you computed them:

(c) (3p) Let p be a positive integer. Compute $\mathbf{A}^p$. Your answer should be a single 3×3 matrix whose entries depend on p. (In other words, please do not give an answer in the form of a product of any matrices.) Describe how you arrived at your answer.

$\mathbf{A}^p$ =

How you determined the formula:

Question 3: Let $\mathbf{x} = [x_1, x_2, x_3]$ be a given nonzero vector in $\mathbb{R}^3$. Define the matrix

$$\mathbf{A_x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

In answering this question, you may assume that $x_3 \neq 0$. No motivation is required for (a) – (d).

(a) (2p) Specify the characteristic polynomial of $\mathbf{A_x}$. (Observe that this will be a polynomial whose coefficients depend on the given vector $\mathbf{x}$.) Specify the real eigenvalues of $\mathbf{A_x}$.

Characteristic polynomial =   Real eigenvalues =

(b) (2p) Specify the kernel of $\mathbf{A_x}$.

Kernel of $\mathbf{A_x}$ =

(c) (1p) Specify the rank of $\mathbf{A_x}$.

Rank of $\mathbf{A_x}$ =

(d) (2p) Given a vector $\mathbf{y} \in \mathbb{R}^3$, set $\mathbf{z} = \mathbf{A_x}\,\mathbf{y}$. Specify $\mathbf{z} \cdot \mathbf{x}$ and $\mathbf{z} \cdot \mathbf{y}$.

$\mathbf{z} \cdot \mathbf{x} =$ $\mathbf{z} \cdot \mathbf{y} =$

Continued on the next page!

(e) (3p) Specify the range of $\mathbf{A_x}$. Prove that your answer is correct.

Range of $\mathbf{A_x}$ =

Motivation:

Hint: Given a vector $\mathbf{y}$*, the vector* $\mathbf{z} = \mathbf{A_x}\,\mathbf{y}$ *is the "vectorial product" or "cross product" of* $\mathbf{x}$ *and* $\mathbf{y}$*. One often writes* $\mathbf{z} = \mathbf{x} \times \mathbf{y}$*. When solving this problem, you can only cite facts that were covered in this course. Do not refer to some property of the vectorial product that you saw elsewhere in your proof for (e).*

Question 4: (10p) Let $\mathcal{P}_r$ denote the vector space of all polynomials of degree at most r, as usual. Consider the following five maps between polynomial spaces:

- T_1 maps $\mathcal{P}_3$ to $\mathcal{P}_2$, according to the formula $[T_1 p](x) = p'(x)$.
- T_2 maps $\mathcal{P}_3$ to $\mathcal{P}_3$, according to the formula $[T_2 p](x) = x\,p'(x) + 1$.
- T_3 maps $\mathcal{P}_2$ to $\mathcal{P}_4$, according to the formula $[T_3 p](x) = x^2\,p(x) + \int_0^x p(y)\,dy$.
- T_4 maps $\mathcal{P}_3$ to $\mathcal{P}_2$, according to the formula $[T_4 p](x) = \dfrac{p(x) - p(0)}{x}$.
- T_5 maps $\mathcal{P}_3$ to $\mathcal{P}_4$, according to the formula $[T_5 p](x) = \left(\dfrac{p(x) - p(0)}{x}\right)^2$.

(a) (5p)For each map T_j check the box that best describes the map.
Note: There should be ONLY ONE check in each column.

	T_1	T_2	T_3	T_4	T_5
T_j is **not** linear:					
T_j is linear and onto but not one-to-one:					
T_j is linear and one-to-one but not onto:					
T_j is linear and neither one-to-one nor onto:					
T_j is linear and both one-to-one and onto:					

(b) (5p) Let $\mathcal{B}_r = (1,\, x,\, x^2,\, \dots,\, x^r)$ denote an ordered bases for $\mathcal{P}_r$. For the maps that were linear, specify the matrix of the map with respect to $\mathcal{B}_r$. (For the maps that are not linear, leave blank!)

The matrix for T_1 is:

The matrix for T_2 is:

The matrix for T_3 is:

The matrix for T_4 is:

The matrix for T_5 is:

Question 5: (10p) Consider the three vectors $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$.

(a) (5p) Use the Gram-Schmidt process to compute an ortho**normal** set $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ such that

- $\text{span}(\mathbf{q}_1) = \text{span}(\mathbf{a}_1)$.
- $\text{span}(\mathbf{q}_1, \mathbf{q}_2) = \text{span}(\mathbf{a}_1, \mathbf{a}_2)$.
- $\text{span}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) = \text{span}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

No motivation is required for (a) and (b).

$\mathbf{q}_1 =$	$\mathbf{q}_2 =$	$\mathbf{q}_3 =$

(b) (3p) Set $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 \, \mathbf{a}_2 \, \mathbf{a}_3 \end{bmatrix}$ and $\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1 \, \mathbf{q}_2 \, \mathbf{q}_3 \end{bmatrix}$. Define $\mathbf{R} = \mathbf{Q}^{\mathrm{T}}\mathbf{A}$, so that $\mathbf{A} = \mathbf{QR}$. Specify $\mathbf{R}$.

$\mathbf{R} =$

(c) (2p) Let m and n denote integers such that $0 < n \leq m$. Let $\{\mathbf{a}_j\}_{j=1}^n$ denote a set of linearly independent vectors in $\mathbb{R}^m$, and let $\{\mathbf{q}_j\}_{j=1}^n$ denote an orthonormal set that results from applying the Gram-Schmidt process to the vectors $\{\mathbf{a}_j\}_{j=1}^n$. Let $\mathbf{A}$ and $\mathbf{Q}$ denote the $m \times n$ matrices whose columns are the vectors $\{\mathbf{a}_j\}_{j=1}^n$ and $\{\mathbf{q}_j\}_{j=1}^n$, respectively. Set $\mathbf{R} = \mathbf{Q}^{\mathrm{T}}\mathbf{A}$. Prove that $\mathbf{R}$ is upper triangular.

You may refer to properties of the Gram-Schmidt process that are listed in Chapter 6 of the book, or mentioned in the lecture without proving them.

Proof: