M341 — Practice problems for first section exam

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Note: This problem sheet is provided to help you study. The choice of topics matches the material covered by the exam, but there was no effort made to try to make it representative in terms of the length or the number of problems.

Problem 1: Motivate your answers briefly.

(a) Let $\mathbf{A}$ and $\mathbf{B}$ be 3×3 matrices, and consider the following four matrices:

$$\begin{aligned}
\mathbf{C} &= \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2, \\
\mathbf{D} &= \mathbf{A}(\mathbf{A}+\mathbf{B}) + \mathbf{B}(\mathbf{B}+\mathbf{A}), \\
\mathbf{E} &= (\mathbf{A}+\mathbf{B})(\mathbf{B}+\mathbf{A}), \\
\mathbf{F} &= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2.
\end{aligned}$$

Which of these matrices *must necessarily* equal $(\mathbf{A}+\mathbf{B})^2$?

(b) Let k be a real number, and consider the matrix

$$\mathbf{A} = \begin{pmatrix} k & 2 \\ 6 & k-1 \end{pmatrix}. \tag{1}$$

For which k is the rank of $\mathbf{A}$ less than 2?

(c) Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be given square matrices of the same size. You know that $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, $\mathbf{C}^{-1}$, $(\mathbf{A}+\mathbf{B})^{-1}$, $(\mathbf{A}+\mathbf{I})^{-1}$, and $(\mathbf{B}+\mathbf{I})^{-1}$ all exist. Solve the equation

$$\mathbf{AXB} + \mathbf{XB} = \mathbf{C}$$

for $\mathbf{X}$. In other words, find an expression $\mathbf{X} = \cdots$ that involves the three given matrices in some combination.

Problem 2: Compute the *inverse* of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Problem 3: Find all solutions to the linear system

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 3x_4 &= 1, \\
x_1 + 2x_2 + 2x_3 + 5x_4 &= 0, \\
-x_1 - 2x_2 - 3x_3 - 7x_4 &= 1.
\end{aligned}$$

Problem 4: Consider the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -5 & 1 & -3 \\ -1 & 1 & 5 & 1 & 3 \\ 1 & 1 & 1 & -1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & -2 & 0 & -2 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrix $\mathbf{B}$ is the RREF of $\mathbf{A}$.

Write down the full solution sets to the following linear systems:

(a) $\left[\begin{array}{rrr|r} 1 & -1 & 1 & -3 \\ -1 & 1 & 1 & 3 \\ 1 & 1 & -1 & 1 \end{array}\right]$

(b) $\left[\begin{array}{rrr|r} 1 & -1 & -5 & 1 \\ -1 & 1 & 5 & 1 \\ 1 & 1 & 1 & -1 \end{array}\right]$

(c) $\left[\begin{array}{rrr|r} 1 & -1 & -5 & -3 \\ -1 & 1 & 5 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right]$

Problem 5: Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^5$. Prove that if $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\|$ then *either* $\mathbf{y} = \mathbf{0}$ *or* $\mathbf{x} \cdot \mathbf{y} \neq 0$.

Note: Be very clear about the logic of your proof. State your assumptions, motivate the steps, etc.

Problem: Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same size. Suppose that $\mathbf{B}$ is invertible. Use induction to prove that for every positive integer n, it is the case that:

$$\left(\mathbf{BAB}^{-1}\right)^n = \mathbf{BA}^n\mathbf{B}^{-1}.$$

Note: Be very clear about the logic of your proof. State your assumptions, motivate the steps, etc.

Problem: Let $\mathbf{x} = [1,\ 0,\ 2,\ -1]$ and $\mathbf{y} = [6,\ 3,\ 1,\ 1]$. Determine $\mathbf{proj}_{\mathbf{x}}\,\mathbf{y}$.