Midterm exam for Numerical Analysis: Linear Algebra

9:00am – 10:45am, Oct. 29, 2019. Closed books.

Question 1: (35p) For this question, please write *only the answer*, no motivation.

(a) Let $\mathbf{A}$ denote an $m \times m$ nonzero matrix for which $\mathbf{A}^2 = \mathbf{A}$. Mark which statements are true: (Where "true" of course means that the statement is *always true* under the given assumptions.)

	TRUE	FALSE
$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) = m$.	TRUE	
If $\|\mathbf{A}\| = 1$, then $\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A})$.	TRUE	
If $\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A})$, then $\|\mathbf{A}\| = 1$.	TRUE	
If $\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A})$, then $\mathbf{A}^* = \mathbf{A}$.	TRUE	

(b) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, let $\alpha \in \mathbb{R}$, and set $\mathbf{A} = \mathbf{I} + \alpha \mathbf{u}\mathbf{v}^*$. For which values of α is $\mathbf{A}$ is invertible?

$\mathbf{A}$ *is invertible whenever* $\alpha \neq -1/\mathbf{v}^*\mathbf{u}$.

(c) Let $\mathbf{A}$ be defined as in problem (b). Provide a formula for $\mathbf{A}^{-1}$ (assuming $\mathbf{A}$ is invertible):

$$\mathbf{A}^{-1} = \mathbf{I} - \frac{\alpha}{1 + \alpha \mathbf{v}^*\mathbf{u}} \mathbf{u}\mathbf{v}^*$$

(b) and (c) were HW problems. Observe that $(\mathbf{I} + \alpha\mathbf{u}\mathbf{v}^*)(\mathbf{I} + \beta\mathbf{u}\mathbf{v}^*) = \mathbf{I} + (\alpha + \beta + \alpha\beta\mathbf{v}^*\mathbf{u})\mathbf{u}\mathbf{v}^*$

(d) Let $\mathbf{A}$ be an $m \times m$ matrix with the SVD $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$. Set $\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^* \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$. Give a formula for an eigenvalue decomposition of $\mathbf{B}$, expressed in terms of the matrices $\mathbf{U}$, $\mathbf{D}$, $\mathbf{V}$.

$\mathbf{B} = \mathbf{W}\mathbf{L}\mathbf{W}^*$ *where* $\mathbf{W} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{V} & -\mathbf{V} \\ \mathbf{U} & \mathbf{U} \end{bmatrix}$ *and* $\mathbf{L} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} \end{bmatrix}$.

(e) Let $\mathbf{u} \in \mathbb{R}^4$ be a non-zero vector and set $\mathbf{A} = \mathbf{I} - \frac{2}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^*$. What are the eigenvalues and singular values of $\mathbf{A}$?

The eigenvalues are $\{1, 1, 1, -1\}$. *The singular values are* $\{1, 1, 1, 1\}$.

Observe that $\mathbf{A}\mathbf{u} = -\mathbf{u}$ *and that* $\mathbf{A}\mathbf{x} = \mathbf{x}$ *for every* $\mathbf{x} \in \langle \mathbf{u} \rangle^\perp$.

(f) Let $\mathbf{A}$ be an $m \times n$ matrix of rank n, where $m > n$. Provide one or two lines of matlab code that produce its *pseudoinverse* $\mathbf{B} = \mathbf{A}^\dagger$. Your answer may not involve the command `pinv`.

Option 1: `B = inv(A'*A)*A';`

Option 2: `[Q,R] = qr(A,0); B = inv(R)*Q';`

Option 3: `[U,D,V] = svd(A,'econ'); B = V*inv(D)*U';`

(g) Specify the following quantities, where the vectors $\mathbf{x}$ range over $\mathbb{C}^m$:

$$\sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}\|_\infty} = \sqrt{m} \qquad \inf_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}\|_\infty} = 1 \qquad \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{x}\|_1}{\|\mathbf{x}\|_2} = \sqrt{m}$$

As usual, $\|\cdot\|_p$ refers to the ℓ^p norm of a vector.

For questions 2 – 5, please motivate all your answers.

Question 2: (10p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

Compute a singular value decomposition of $\mathbf{A}$ (either the economy or the full SVD).

Solution: Observe that the columns of $\mathbf{A}$ are already orthogonal, so if we just normalize them, we get the decomposition

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$$

where the left factor is an orthonormal matrix. This is *almost* an SVD of $\mathbf{A}$ (with the matrix of right singular vectors being $\mathbf{I}$), we just need to reorder the singular values so that they decay in magnitude:

$$\mathbf{A} = \underbrace{\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}}_{=\mathbf{U}} \underbrace{\begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{3} \end{bmatrix}}_{=\mathbf{D}} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{=\mathbf{V}^*}$$

Question 3: (20p) Consider the function $f(x) = 1 - \cos(x)$ as a function of x from $\mathbb{R}$ to $\mathbb{R}$.

(a) Compute the relative condition number $\kappa_f(x)$. Is f well-conditioned for every x? If not, then specify where the potentially problematic locations are.

(b) Set $\beta = 10^{-10}$ and estimate $f(\beta)$ to at least fifteen correct digits of accuracy using a Taylor expansion.

(c) What would be the output of the matlab command "`f = 1 - cos(1e-10)`"? How many correct digits would you get? (Assume standard double precision accuracy, so that $\epsilon_{\text{mach}} \approx 10^{-16}$.)

(d) Give a matlab command that would accurately evaluate $f(x)$ for $x \in [-1, 1]$. (There is no need to prove anything, just provide the command.)

Solution:

(a) Using a standard formula for the condition number of a differentiable function, we get

$$\kappa_f(x) = \left|\frac{f'(x)}{f(x)/x}\right| = \left|\frac{\sin(x)\, x}{1 - \cos(x)}\right|.$$

For x small, we use a Taylor expansion to see that

$$\kappa_f(x) = \left|\frac{(x + O(x^3))\, x}{(1/2)x^2 + O(x^4)}\right| = 2 + O(x^2)$$

so f is well-conditioned near $x = 0$.

However, at points other than zero, the condition number goes to infinity whenever $1 - \cos(x) \to 0$.

Answer: The condition number of f goes to infinity when $x \to 2\pi n$ for any integer n except 0.

(b) As $x \to 0$, we have

$$f(x) = 1 - \big(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)\big) = \frac{1}{2}x^2 - \frac{1}{24}x^4 + O(x^6).$$

Since $(\beta^4/24)/(\beta^2/2) = \beta^2/12 \ll 10^{-15}$, we can ignore the x^4 term, and find

$$f(\beta) \approx \frac{1}{2}\beta^2 = 0.5 \times 10^{-20}.$$

(c) Since $\cos(\beta) \approx 1 - (1/2)\beta^2 = 1 - (1/2)\,10^{-20}$, we see that when $\cos(\beta)$ is evaluated in floating point arithmetic, the answer will be 1. Consequently, `1-cos(1e-10)` will evaluate to zero, which would represent an error of size $O(1)$ (relative to the exact value of $f(\beta)$). No accurate digits.

(To be absolutely strict, all we could say for sure is that `cos(1e-10)` evaluates to something within machine precision of 1, not necessarily exactly one. But this does not change the answer; in fact this would result in a relative error far larger than $O(1)$.)

(d) To avoid the subtraction of two large numbers that are very close to each other, we could for instance use the trig identity

$$1 - \cos(x) = 1 - \cos(x/2)^2 + \sin(x/2)^2 = \sin(x/2)^2 + \sin(x/2)^2 = 2\sin(x/2)^2,$$

which would lead to the answer: `f = 2*(sin(1e-10/2)^ 2)`

Alternatively, you could use $1 - \cos(x) = \dfrac{(1 - \cos(x))(1 + \cos(x))}{1 + \cos(x)} = \dfrac{1 - \cos(x)^2}{1 + \cos(x)} = \dfrac{\sin(x)^2}{1 + \cos(x)}.$

Question 4: (15p) Let $\mathbf{A}$ be an $m \times m$ matrix of rank k, where $k < m$. Prove that

$$\|\mathbf{A}\|_{\text{Fro}} \leq \sqrt{k}\|\mathbf{A}\|,$$

where $\|\cdot\|$ denotes the spectral norm, and $\|\cdot\|_{\text{Fro}}$ denotes the Frobenius norm.

Solution: Let $\{\sigma_j\}_{j=1}^{\min(m,n)}$ denote the singular values of $\mathbf{A}$. Recall that

$$\|\mathbf{A}\| = \sigma_1 \qquad \text{and} \qquad \|\mathbf{A}\|_{\text{Fro}} = \left(\sum_{j=1}^{\min(m,n)} \sigma_j^2\right)^{1/2}.$$

When the rank of $\mathbf{A}$ is k, $\sigma_j = 0$ whenever $j > k$, so

$$\|\mathbf{A}\|_{\text{Fro}} = \left(\sum_{j=1}^{k} \sigma_j^2\right)^{1/2} \leq \{\text{Use } \sigma_j \leq \sigma_1 \text{ for all } j.\} \leq \left(\sum_{j=1}^{k} \sigma_1^2\right)^{1/2} = \left(k\sigma_1^2\right)^{1/2} = \sqrt{k}\sigma_1 = \sqrt{k}\|\mathbf{A}\|,$$

which completes the proof.

Question 5: (20p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Perform by hand *one step* of the Householder QR factorization procedure on $\mathbf{A}$. In other words, build a unitary matrix $\mathbf{Q}$ such that the matrix $\mathbf{B} = \mathbf{Q}^*\mathbf{A}$ takes the form

$$\mathbf{B} = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}.$$

Your answer should specify both $\mathbf{Q}$ and $\mathbf{B}$.

Solution: Let $\mathbf{a} = [1, 1, 1, 1]^{\rm t}$ denote the first column of $\mathbf{A}$. Then the first Householder vector is

$$\mathbf{v} = \|\mathbf{a}\|\mathbf{e}_1 - \mathbf{a} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

This leads to the Householder reflector

$$\mathbf{Q} = \mathbf{I} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^* = \mathbf{I} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Finally,

$$\mathbf{B} = \mathbf{Q}^*\mathbf{A} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note: If you use the other choice of $\mathbf{v}$, you get

$$\mathbf{v} = -\|\mathbf{a}\|\mathbf{e}_1 - \mathbf{a} = -2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ -1 \\ -1 \end{bmatrix}.$$

This leads to the Householder reflector

$$\mathbf{Q} = \mathbf{I} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^* = \mathbf{I} - \frac{1}{6}\begin{bmatrix} -3 \\ -1 \\ -1 \\ -1 \end{bmatrix}\begin{bmatrix} -3 & -1 & -1 & -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -3 & -3 & -3 & -3 \\ -3 & 5 & -1 & -1 \\ -3 & -1 & 5 & -1 \\ -3 & -1 & -1 & 5 \end{bmatrix}.$$

Finally,

$$\mathbf{B} = \mathbf{Q}^*\mathbf{A} = \begin{bmatrix} -2 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$