Midterm exam for Numerical Analysis: Linear Algebra

9:00am – 10:45am, Oct. 29, 2019. Closed books.

Question 1: (35p) For this question, please write *only the answer*, no motivation.

(a) Let $\mathbf{A}$ denote an $m \times m$ nonzero matrix for which $\mathbf{A}^2 = \mathbf{A}$. Mark which statements are true: (Where "true" of course means that the statement is *always true* under the given assumptions.)

	TRUE	FALSE
$\text{rank}(\mathbf{A}) + \text{rank}(\mathbf{I} - \mathbf{A}) = m$.		
If $\|\mathbf{A}\| = 1$, then $\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A})$.		
If $\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A})$, then $\|\mathbf{A}\| = 1$.		
If $\text{range}(\mathbf{A}) \perp \text{null}(\mathbf{A})$, then $\mathbf{A}^* = \mathbf{A}$.		

(b) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$, let $\alpha \in \mathbb{R}$, and set $\mathbf{A} = \mathbf{I} + \alpha \mathbf{u}\mathbf{v}^*$. For which values of α is $\mathbf{A}$ is invertible?

(c) Let $\mathbf{A}$ be defined as in problem (b). Provide a formula for $\mathbf{A}^{-1}$ (assuming $\mathbf{A}$ is invertible):

(d) Let $\mathbf{A}$ be an $m \times m$ matrix with the SVD $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^*$. Set $\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^* \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$. Give a formula for an eigenvalue decomposition of $\mathbf{B}$, expressed in terms of the matrices $\mathbf{U}$, $\mathbf{D}$, $\mathbf{V}$.

(e) Let $\mathbf{u} \in \mathbb{R}^4$ be a non-zero vector and set $\mathbf{A} = \mathbf{I} - \frac{2}{\|\mathbf{u}\|^2}\mathbf{u}\mathbf{u}^*$. What are the eigenvalues and singular values of $\mathbf{A}$?

(f) Let $\mathbf{A}$ be an $m \times n$ matrix of rank n, where $m > n$. Provide one or two lines of matlab code that produce its *pseudoinverse* $\mathbf{B} = \mathbf{A}^\dagger$. Your answer may not involve the command `pinv`.

(g) Specify the following quantities, where the vectors $\mathbf{x}$ range over $\mathbb{C}^m$:

$$\sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}\|_\infty} = \qquad \inf_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{x}\|_2}{\|\mathbf{x}\|_\infty} = \qquad \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{x}\|_1}{\|\mathbf{x}\|_2} =$$

As usual, $\|\cdot\|_p$ refers to the ℓ^p norm of a vector.

For questions 2 – 5, please motivate all your answers.

Question 2: (10p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -2 \end{bmatrix}.$$

Compute a singular value decomposition of $\mathbf{A}$ (either the economy or the full SVD).

Question 3: (20p) Consider the function $f(x) = 1 - \cos(x)$ as a function of x from $\mathbb{R}$ to $\mathbb{R}$.

(a) Compute the relative condition number $\kappa_f(x)$. Is f well-conditioned for every x? If not, then specify where the potentially problematic locations are.

(b) Set $\beta = 10^{-10}$ and estimate $f(\beta)$ to at least fifteen correct digits of accuracy using a Taylor expansion.

(c) What would be the output of the matlab command "`f = 1 - cos(1e-10)`"? How many correct digits would you get? (Assume standard double precision accuracy, so that $\epsilon_{\text{mach}} \approx 10^{-16}$.)

(d) Give a matlab command that would accurately evaluate $f(x)$ for $x \in [-1, 1]$. (There is no need to prove anything, just provide the command.)

Question 4: (15p) Let $\mathbf{A}$ be an $m \times m$ matrix of rank k, where $k < m$. Prove that

$$\|\mathbf{A}\|_{\text{Fro}} \leq \sqrt{k}\|\mathbf{A}\|,$$

where $\|\cdot\|$ denotes the spectral norm, and $\|\cdot\|_{\text{Fro}}$ denotes the Frobenius norm.

Question 5: (20p) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Perform by hand *one step* of the Householder QR factorization procedure on $\mathbf{A}$. In other words, build a unitary matrix $\mathbf{Q}$ such that the matrix $\mathbf{B} = \mathbf{Q}^*\mathbf{A}$ takes the form

$$\mathbf{B} = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}.$$

Your answer should specify both $\mathbf{Q}$ and $\mathbf{B}$.

Hint: Recall that there are two possible Householder reflectors to choose from. In this problem, feel free to choose either one (no need to worry about round off errors).