BASIC CONCEPTS:

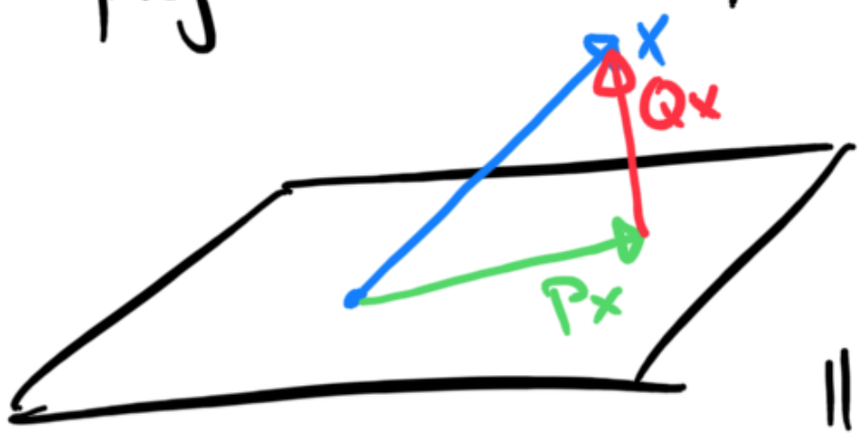
* Inner product, outer product $x \cdot y = x^* y$
* Linear independence, bases, ON bases.
* Fundamental subspaces, col/row space, null space.
* Rank
* Norms of vectors, Euclidean, l^p norms.
* Norms of matrices.
  Some are operator norms $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$
  Not every matrix norm is an operator norm.
* Unitary matrices, ON matrices, $\|Ux\| = \|x\|$ if U is ON
* For spectral norm of a matrix: $\|A\| = \|UA\| = \|AV\|$ if U & V are unitary
* Cauchy Schwartz

SVD VERY IMPORTANT! (2)

* Every matrix has an SVD.
* Can be used to solve linear systems, least squares problems, low rank approximation, ...
* ON-bases for fundamental subspaces.
* Economy size versus full.
* Geometry - hyperellipses
* EVD vs SVD: $A = U \Sigma V^*$ (m×n) $\Rightarrow$ $AA^* = U \Sigma^2 U^*$ (m×m), $A^*A = V \Sigma^2 V^*$ (n×n)
* EVD is better for powers of A
  $A = V \Lambda V^{-1} \Rightarrow p(A) = V p(\Lambda) V^{-1}$
* Spectral norm $= \sigma_1$ Frob norm $= \left(\sum_j \sigma_j^2 \right)^{1/2}$
* Eckart-Young thm.

PROJECTIONS

* $P^2 = P$
* $M = \text{ran}(P)$ $\quad N = \text{null}(P)$ $\quad V = M \oplus N$.
* orthog. proj^n : $P^* = P$, $\text{ran}(P) = \ker(P)^\perp$, $\|P\| = 1$



$Qx = (I - P)x$

$Px \perp Qx$

$\|x\|^2 = \|Px\|^2 + \|Qx\|^2$

$\|x - Px\| = \inf_{y \in M} \|x - y\|$

* Let A be a matrix. $M = \text{col}(A)$.
  P = orthog proj^n onto $M = A(A^*A)^{-1}A^* = QQ^* = UU^*$

from SVD

Q from QR

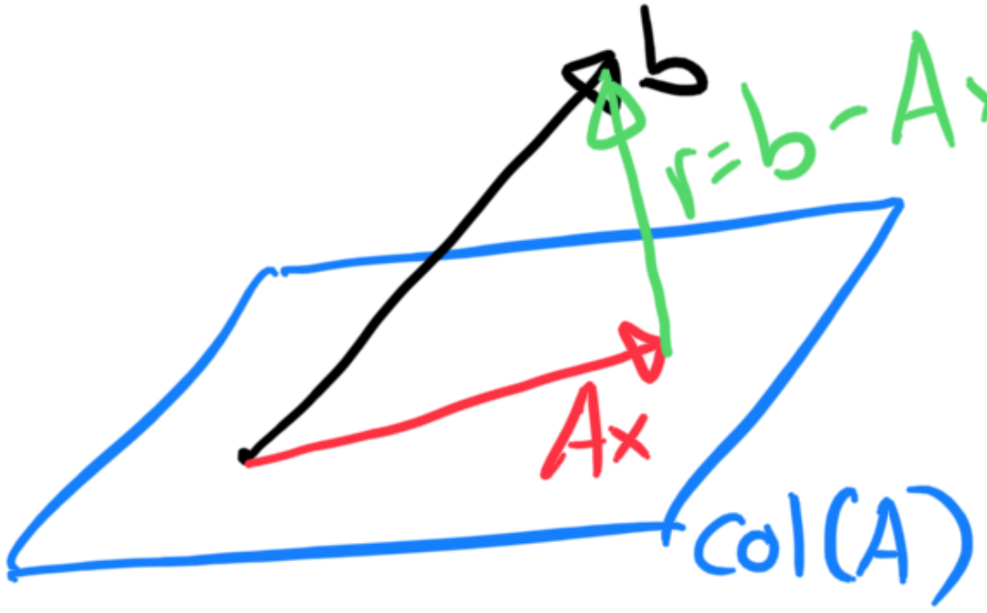
QR FACTORIZATION

$A = QR$

* Exists for every matrix
* Methods for computing:
  - GRAM-SCHMIDT (classical)
  - MODIFIED G-S
  - **HOUSEHOLDER**
* How to use it:
  - Solving linear systems
  - Least square problems.
  - Finding projectors
  - Basis for range.

LEAST SQUARES PROBLEMS

* Definition of soln to $Ax=b$
* Geometry
* How to compute soln
  - SVD
  - QR
  - Normal eqns ← ill-cond!



STABILITY & CONDITIONING

* Floating point arithmetic
* Round off errors. Def" of ϵ_{mach}
* Condition number
* Forwards & Backwards stability
* Concept of "significant digits"
* Summation: Order by magnitude.
* Conditioning of least squares problems

LU FACTORIZATION & GAUSSIAN ELIM (7)

* Connection between G.E. and LU factorization.
* LU is cheaper than QR/SVD for solving linear systems.
* Pivoting.
* Stability of LU.
* (Better for sparse matrices)
* CHOLESKY $A = R^* R$ FOR SPD matrices only!

EIGENVALUE PROBLEMS

* Square matrices only.
* $Av = \lambda v$ for $v \neq 0$
* $p(x) = \det(xI - A)$
* Geometric multiplicity
  vs. Algebraic ——"——
* Defective matrices.
* $A = V \Lambda V^{-1}$ ← Does not always exist!
  ↑ diagonal
* If $A = A^*$ (or $AA^* = A^*A$ more generally)
  then $A = U \Lambda U^*$ with U unitary
* Schur decomposition
  $A = U T U^*$ Always exists!
  ↑ triangular
* $A^k = V \Lambda^k V^{-1}$ $\exp(A) = V \exp(\Lambda) V^{-1}$
* Similarity transform: $B = XAX^{-1}$

ALGORITHMS FOR EVALS AND EVECS

* Must be iterative if $m \geq 5$
* Power iteration. Shifted inverse iteration.
  → Speed of convergence
* Rayleigh quotient → Cubic convergence when $A = A^*$!
* Block power iteration.
* Tridiagonal form (when $A = A^*$)
  Hessenberg form
* QR algorithm

KRYLOV METHODS

* Defn of Krylov space

$K_n = \text{span}(b, Ab, A^2b, \ldots, A^{n-1}b)$

* Uses only matrix-vector multiply ←

  Good for large sparse matrices

* Arnoldi: $Ax=\lambda x$ Build $A = QHQ^*$ one column at a time

$$AQ_n = Q_{n+1}\tilde{H}_n$$

* GMRES $Ax=b$

* Lanczos $Ax=\lambda x$ symmetric matrices 3-term recurrance

* Conjugate gradients $Ax=b$ - for SPD matrices

Final exam:

- Saturday Dec 14, 19:00 - 22:00. Room: BUR 136.

- Cumulative exam. All material covered in the class is included.

- Material covered after the midterm will be emphasized *slightly* heavier.

- The first question will involve several short questions based on the homeworks. Use your time wisely.