ARNOLDI

$A \in \mathbb{C}^{m \times m}$ $b \in \mathbb{C}^{m \times 1}$

Build $A = QHQ^*$ vector by vector

b = starting vec.

$q_1 = \frac{1}{\|b\|} b$

FOR $j = 1, 2, 3, \ldots$

  $v = A q_j$

  FOR $i = 1, 2, 3, \ldots, j$

    $h_{ij} = q_i^* v$

    $v = v - h_{ij} q_i$

  END

  $h_{j+1,j} = \|v\|$

  $q_{j+1} = \frac{1}{h_{j+1,j}} v$

END

Evals of $H(1:n, 1:n) \approx$ "some" evals of A

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$A \in \mathbb{C}^{m \times m}$ $\quad b \in \mathbb{C}^{m \times 1}$

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FOR $j = 1, 2, 3, \ldots$

$\quad v = A q_j$

$\quad$ FOR $i = 1, 2, 3, \ldots, j$

$\quad\quad h_{ij} = q_i^* v$ ← mostly zero since H is tridiag

$\quad\quad v = v - h_{ij} q_i$

$\quad$ END

$\quad h_{j+1,j} = \|v\|$

$\quad q_{j+1} = \frac{1}{h_{j+1,j}} v$

END

Evals of $H(1:n, 1:n) \approx$ "some" evals of A

Suppose $A = A^+$ is real!

$H = H^+ \Rightarrow H = T$ is tridiag

$A = QTQ^*$

Just two steps involve actual work!

②

LANCZOS

Given real sym $A \in \mathbb{R}^{m \times m}$

$b \in \mathbb{R}^{m \times 1}$ is starting vector

Seek $A = Q T Q^*$

Different notation!

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & - \\ \beta_1 & \alpha_2 & \beta_2 & 0 & - \\ 0 & \beta_2 & \alpha_3 & \beta_4 & - \\ 0 & 0 & \beta_3 & \alpha_4 & - \\ | & | & | & | & \end{bmatrix}$$

$\beta_0 = 0 \quad q_0 = 0 \quad q_1 = \frac{1}{\|b\|} b$

FOR $j = 1, 2, 3, \ldots$

$\quad v = A q_j$

$\quad \alpha_j = q_j^t v$

$\quad v = v - \beta_{j-1} q_{j-1} - \alpha_j q_j$

$\quad \beta_j = \|v\|$

$\quad q_{j+1} = \frac{1}{\beta_j} v$

END

* Only need to store 3 vectors in order to compute T.
* May lose orthogonality due to round-off errors. → Reorthogonalize
* Sometimes converges <u>very</u> fast!

KRYLOV METHODS

* For general square matrix: Arnoldi iteration
  - → Approximations to evals
  - → Solve $Ax=b$ using GMRES.
* For real symmetric matrix:
  Arnoldi becomes Lanczos.
  Faster due to 3-term recurrence.
  Better convergence.
  - → Solve $Ax=b$ using MINRES ← Not covered
* If A is real and positive, then method of choice is CONJUGATE GRADIENTS (CG)
* There are specialized methods for general A.
  E.g. work with $B = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$
* Preconditioners.

VIDEO LECTURES ONLINE! PURELY VOLUNTARY...