Homework set 5 — CSE 383C / CS 383C / M 383E / ME 397, Fall 2019

Hand in solutions to: 17.1, 18.1, 19.1, from the book.

Problem 1: Suppose that you are working on a computer that satisfies axiom (13.7) from the book *for real valued x and y*, so that for any $x, y \in \mathbf{F}$, we have

$$x \otimes y = (x * y)\,(1 + \epsilon)$$

for some ϵ such that $|\epsilon| \leqslant \epsilon_{\text{mach}}$. Suppose now that you use the real valued floating point arithmetic to define complex floating point arithmetic. For instance for $x_1, x_2, y_1, y_2 \in \mathbf{F}$, you define complex numbers $x = x_1 + ix_2$ and $y = y_1 + iy_2$, and then complex multiplication via

$$m(x, y) = \big((x_1 \otimes y_1) \ominus (x_2 \otimes y_2)\big) \oplus i\big((x_1 \otimes y_2) \oplus (x_2 \otimes y_1)\big).$$

Prove that

$$m(x, y) = xy(1 + \delta)$$

for some δ such that

$$|\delta| \leqslant C\epsilon_{\text{mach}} + O(\epsilon_{\text{mach}}^2).$$

You do not need to get the optimal value of C. (But there is of course also no need to be wasteful!)

Problem 2 (optional – do not hand in): Consider the same setup as in Problem 1, but now for the complex division:

$$d(x, y) \;=\; \Big(\big((x_1 \otimes y_1) \oplus (x_2 \otimes y_2)\big) \oplus i\big((x_2 \otimes y_1) \ominus (x_1 \otimes y_2)\big)\Big) \mathbin{\dot{\div}} \big((y_1 \otimes y_1) \oplus (y_2 \otimes y_2)\big).$$

What can you say about the stability of d?

Problem 3: Consider the evaluation of the complex multiplication

$$z = (a + ib)(c + id) = (ac - bd) + i(ad + bc). \tag{1}$$

Observe that formula (1) is mathematically equivalent to

$$z = (ac - bd) + i\big[(a + b)(c + d) - ac - bd\big]. \tag{2}$$

The point here is that (1) involves 4 multiplications, while (2) involves 3 multiplications. This is a minor saving for scalars, but does make a difference for matrices. Suppose that $a, b, c, d \in \mathbb{R}^{m \times m}$, and work out home many floating point additions and multiplications (count them separately) are required to evaluate the product $z = (a + ib)(c + id)$ using formulas (1) and (2), respectively.

Problem 4 (optional — do not hand in): Is there a difference in numerical stability between (1) and (2)? It may be helpful to consider what happens if $a = c = \theta$ and $d = b = 1/\theta$ for a real number θ that is very large.

P.G. Martinsson, UT-Austin, Oct. 2019