Final exam for Numerical Analysis: Linear Algebra

December 14, 2019. Closed books.

Hand in solutions on separate sheets. Motivate all answers (except Question 1). Write your name!

Question 1: (25p) For this question, please provide *only the answer*, no motivation.

(a) Consider the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where $\mathbf{A}_{11}$ is a square and non-singular. Then $\mathbf{A}$ admits the factorization $\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{X} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}$. Specify the matrices $\mathbf{X}$ and $\mathbf{Y}$.

$\mathbf{X} = \mathbf{A}_{21}\mathbf{A}_{11}^{-1}$ $\qquad$ $\mathbf{Y} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$

(b) Let $\mathbf{A}$ be a real square matrix for which $\|\mathbf{Ax}\| = \|\mathbf{x}\|$ for every vector $\mathbf{x}$. What can you say about the singular values of $\mathbf{A}$? What about the eigenvalues?

Every singular value is 1. Every eigenvalue has modulus 1.

(c) Let $\mathbf{A}$ be a 5×5 square matrix with the characteristic polynomial $p(x) = \det(x\mathbf{I} - \mathbf{A})$. Suppose that $\mathbf{A}^3 = \mathbf{0}$. What does this tell you about p?

$p(x) = x^5$.

(Observe that if $\mathbf{Av} = \lambda\mathbf{v}$, then $\mathbf{A}^3\mathbf{v} = \lambda^3\mathbf{v}$. Since $\mathbf{A}^3 = 0$, the only possible eigenvalue is 0.)

(d) Let $\mathbf{A} \in \mathbb{C}^{5\times5}$ and $\mathbf{b} \in \mathbb{C}^{5\times1}$. Suppose that the Krylov spaces $\mathcal{K}_p = \text{span}(\mathbf{b}, \mathbf{Ab}, \dots, \mathbf{A}^{p-1}\mathbf{b})$ have dimension precisely p for $p = 1, 2, \dots, 5$. Let $\{\mathbf{q}_j\}_{j=1}^5$ be a sequence of vectors such that $\{\mathbf{q}_j\}_{j=1}^p$ forms an orthonormal basis for $\mathcal{K}_p$ for $p = 1, 2, \dots, 5$. Which of the following statements are true:

(i) If $i < j$, then $\mathbf{q}_i^*\mathbf{A}\mathbf{q}_j = 0$. FALSE

(ii) If $i > j+1$, then $\mathbf{q}_i^*\mathbf{A}\mathbf{q}_j = 0$. TRUE

(iii) If $\mathbf{A}$ is symmetric and $i < j-1$, then $\mathbf{q}_i^*\mathbf{A}\mathbf{q}_j = 0$. TRUE

(iv) The set $\{\mathbf{A}^2\mathbf{b}, \mathbf{A}^3\mathbf{b}, \mathbf{A}^4\mathbf{b}\}$ is linearly independent. TRUE

(e) Let $\mathbf{A}$ be a 3×3 matrix of rank 2. Let $\mathbf{b} = [1, 2, 1]^*$, and let $\mathbf{x}_\star$ denote the least squares solution to $\mathbf{Ax} = \mathbf{b}$. You know that $\mathbf{u}_1 \cdot \mathbf{b} = 1$ and $\mathbf{u}_2 \cdot \mathbf{b} = \sqrt{2}$, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the two left singular vectors of $\mathbf{A}$ associated with non-zero singular values. Evaluate $\|\mathbf{Ax}_\star - \mathbf{b}\|$.

$\|\mathbf{Ax}_\star - \mathbf{b}\| = \sqrt{3}$.

To see why, recall that $\mathbf{Ax}_\star = \mathbf{Pb}$ where $\mathbf{P}$ is the orthogonal projection onto the range of $\mathbf{A}$. Moreover, $\|\mathbf{b}\|^2 = \|\mathbf{Ax}_\star - \mathbf{b}\|^2 + \|\mathbf{Pb}\|^2$ by orthogonality. We have $\|\mathbf{Pb}\|^2 = \|\mathbf{u}_1(\mathbf{u}_1 \cdot \mathbf{b}) + \mathbf{u}_2(\mathbf{u}_2 \cdot \mathbf{b})\|^2 = |\mathbf{u}_1 \cdot \mathbf{b}|^2 + |\mathbf{u}_2 \cdot \mathbf{b}|^2 = 1 + 2 = 3$. Finally, $\|\mathbf{Ax}_\star - \mathbf{b}\|^2 = \|\mathbf{b}\|^2 - \|\mathbf{Pb}\|^2 = 6 - 3 = 3$.

Question 2: (15p) Consider the two vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

(a) (5p) Apply the Gram-Schmidt process to the vectors $\mathbf{u}$ and $\mathbf{v}$, in that order, to produce two orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$. Specify $\mathbf{x}$ and $\mathbf{y}$.

(b) (5p) Set $\mathbf{A} = \begin{bmatrix} \mathbf{u} \ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 2 & -2 \end{bmatrix}$. Compute the QR factorization of $\mathbf{A}$.

(c) (5p) Set $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ -6 & -1 \\ 6 & 2 \end{bmatrix}$. Compute the QR factorization of $\mathbf{B}$.

Solution:

(a) Set $r_{11} = \|\mathbf{u}\| = 3$. Then $\mathbf{x} = \frac{1}{r_{11}}\mathbf{u} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

Set $r_{12} = \mathbf{x} \cdot \mathbf{v} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \frac{1}{3}(-6) = -2.$

Set $\mathbf{w} = \mathbf{v} - r_{12}\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} - (-2)\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}$.

Set $r_{22} = \|\mathbf{w}\| = \frac{1}{3}\sqrt{4+1+4} = 1.$

Finally, $\mathbf{y} = \frac{1}{r_{22}}\mathbf{w} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}$.

(b) Simply observe that $\mathbf{Q} = \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & -1/3 \\ 2/3 & -2/3 \end{bmatrix}$ and that $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$.

(c) The columns of $\mathbf{B}$ are the same as the columns of $\mathbf{A}$, but scaled: $\mathbf{B} = \mathbf{A}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$.

It follows that the QR factorization of $\mathbf{B}$ is given as

$$\mathbf{B} = \mathbf{A}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{QR}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ -2/3 & -1/3 \\ 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ 0 & -1 \end{bmatrix}.$$

Note: In the solution to (a), I normalized $\mathbf{x}$ and $\mathbf{y}$ to have length 1. This was not required, so you got full points if your answer consists of some scaled versions. But in (b), you do need to rescale when you form the QR factorization since the columns of $\mathbf{Q}$ must have unit length.

Question 3: (15p) Consider the matrices $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 3 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 & 3 \\ 4 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

(a) (7p) Find a unitary matrix $\mathbf{Q}$ and a tridiagonal matrix $\mathbf{H}$ such that $\mathbf{A} = \mathbf{QHQ}^*$.

(b) (8p) Find a unitary matrix $\mathbf{Q}$ and a tridiagonal matrix $\mathbf{H}$ such that $\mathbf{B} = \mathbf{QHQ}^*$.

Solution:

(a) All we need to do is to swap the second and the third rows, and then apply the analogous operations to the columns. The permutation matrix is $\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and we find that

$$\mathbf{H} = \mathbf{Q}^*\mathbf{AQ} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 7 & 3 \\ 0 & 3 & 2 \end{bmatrix}.$$

(b) Solution based on Householder reflectors: We seek a matrix $\mathbf{Q}$ of the form

$$\mathbf{Q} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix}$$

where $\mathbf{L}$ is the 2×2 Householder reflector for which $\mathbf{L}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{4^2+3^2} \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Set $\mathbf{z} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. Then the Householder vector is $\mathbf{v} = \|\mathbf{z}\|\mathbf{e}_1 - \mathbf{z} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$.
This results in the Householder reflector

$$\mathbf{L} = \mathbf{I} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{vv}^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/5 & -3/5 \\ -3/5 & 9/5 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}.$$

Multiplying the matrices together, we get $\mathbf{H} = \mathbf{Q}^*\mathbf{BQ} = \begin{bmatrix} 1 & 5 & 0 \\ 5 & 49/25 & -7/25 \\ 0 & -7/25 & 1/25 \end{bmatrix}$.

Note: You could of course also use the vector $\mathbf{v} = -\|\mathbf{z}\|\mathbf{e}_1 - \mathbf{z} = -\begin{bmatrix} 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ -3 \end{bmatrix}$.

Then $\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4/5 & -3/5 \\ 0 & -3/5 & 4/5 \end{bmatrix}$ and $\mathbf{H} = \begin{bmatrix} 1 & -5 & 0 \\ -5 & 49/25 & -7/25 \\ 0 & -7/25 & 1/25 \end{bmatrix}$.

(b) Solution based on Givens rotations. We seek a Givens rotation $\mathbf{G} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ such that

$$\mathbf{G}^*\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \sqrt{4^2+3^2} \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

One easily finds that $\mathbf{G} = \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix}$. Setting

$$\mathbf{Q} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix}$$

we get

$$\mathbf{H} = \mathbf{Q}^*\mathbf{BQ} = \begin{bmatrix} 1 & 5 & 0 \\ 5 & 49/25 & 7/25 \\ 0 & 7/25 & 1/25 \end{bmatrix}.$$

Question 4: (15p) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. Recall that we say that $\mathbf{A}$ is *symmetric positive definite (spd)* if $\mathbf{A}$ is symmmetric, and for every nonzero vector $\mathbf{v} \in \mathbb{R}^{m\times 1}$ it is the case that $\mathbf{v}^*\mathbf{A}\mathbf{v} > 0$.

(a) (5p) Suppose that $\mathbf{A} = \mathbf{X}^*\mathbf{X}$ for some nonsingular square matrix $\mathbf{X}$. Prove that $\mathbf{A}$ is spd.

(b) (4p) Suppose that $\mathbf{A} = \mathbf{R}^*\mathbf{R}$ where $\mathbf{R}$ is an upper triangular matrix. What is $\mathbf{R}(1,1)$?

(c) (6p) Compute the Cholesky factorization of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution: (a) First we verify that $\mathbf{A}$ is symmetric: $\mathbf{A}^* = (\mathbf{X}^*\mathbf{X})^* = \mathbf{X}^*\mathbf{X}^{**} = \mathbf{X}^*\mathbf{X} = \mathbf{A}$.

Fix $\mathbf{v} \neq \mathbf{0}$. Then

$$\mathbf{v}^*\mathbf{A}\mathbf{v} = \mathbf{v}^*\mathbf{X}^*\mathbf{X}\mathbf{v} = (\mathbf{X}\mathbf{v})^*\mathbf{X}\mathbf{v} = \|\mathbf{X}\mathbf{v}\|^2.$$

Since $\mathbf{X}$ is nonsingular, we know that $\mathbf{X}\mathbf{v} \neq \mathbf{0}$, so $\|\mathbf{X}\mathbf{v}\|^2 > 0$.

(b) Since $\mathbf{R}$ is upper-triangular, we immediately see that $\mathbf{A} = \mathbf{R}^*\mathbf{R}$ implies that $\mathbf{A}(1,1) = \mathbf{R}(1,1)\mathbf{R}(1,1)$. It follows that $\mathbf{R}(1,1) = \sqrt{\mathbf{A}(1,1)}$.

(c) Set $\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$$\mathbf{A}_2 := \mathbf{R}_1^{-*}\mathbf{A}\mathbf{R}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

In the next step, we have $\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ which leads to

$$\mathbf{A}_3 := \mathbf{R}_2^{-*}\mathbf{A}_2\mathbf{R}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It follows that

$$\mathbf{R} = \mathbf{R}_1\mathbf{R}_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 5: (15p) Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ be a non-singular matrix, and let $\mathbf{b} \in \mathbb{C}^m$ be a fixed vector. Suppose that you have an algorithm for solving $\mathbf{A}\mathbf{x} = \mathbf{b}$ that produces an answer $\tilde{\mathbf{x}}$ that satisfies $(\mathbf{A} + \delta\mathbf{A})\tilde{\mathbf{x}} = \mathbf{b}$ for some matrix $\delta\mathbf{A}$ such that

$$\frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|} \leq \varepsilon. \tag{1}$$

(a) (10p) Prove that

$$\frac{\|\tilde{\mathbf{x}} - \mathbf{x}_\star\|}{\|\tilde{\mathbf{x}}\|} \leq \kappa(\mathbf{A})\,\varepsilon, \tag{2}$$

where $\mathbf{x}_\star$ is the exact solution to $\mathbf{A}\mathbf{x}_\star = \mathbf{b}$, and where $\kappa(\mathbf{A})$ is the condition number of $\mathbf{A}$. Observe that (2) looks slightly different from the bounds we saw in class, since it has $\|\tilde{\mathbf{x}}\|$ in the denominator instead of $\|\mathbf{x}_\star\|$. This change permits us to get a bound that holds for every positive ε, not just asymptotically as $\varepsilon \to 0$.

(b) (5p) Suppose that (1) holds asymptotically as the rounding precision $\varepsilon \to 0$. Prove that

$$\frac{\|\tilde{\mathbf{x}} - \mathbf{x}_\star\|}{\|\mathbf{x}_\star\|} \leq \kappa(\mathbf{A})\,\varepsilon + O(\varepsilon^2), \qquad \text{as } \varepsilon \to 0. \tag{3}$$

Solution:

(a) We find that

$$\tilde{\mathbf{x}} - \mathbf{x}_\star = \mathbf{A}^{-1}\big(\mathbf{A}\tilde{\mathbf{x}} - \mathbf{A}\mathbf{x}_\star\big) \overset{(A)}{=} \mathbf{A}^{-1}\big(\mathbf{b} - (\delta\mathbf{A})\tilde{\mathbf{x}} - \mathbf{A}\mathbf{x}_\star\big) \overset{(B)}{=} -\mathbf{A}^{-1}(\delta\mathbf{A})\tilde{\mathbf{x}}.$$

In step (A), we use that $(\mathbf{A} + \delta\mathbf{A})\tilde{\mathbf{x}} = \mathbf{b}$. In step (B), we use that $\mathbf{A}\mathbf{x}_\star = \mathbf{b}$. Take norms to get

$$\|\tilde{\mathbf{x}} - \mathbf{x}_\star\| \leq \|\mathbf{A}^{-1}\|\,\|\delta\mathbf{A}\|\,\|\tilde{\mathbf{x}}\| \overset{(C)}{\leq} \|\mathbf{A}^{-1}\|\,\|\mathbf{A}\|\,\varepsilon\,\|\tilde{\mathbf{x}}\|.$$

In step (C), we used (1). Divide by $\|\tilde{\mathbf{x}}\|$ and use the definition $\kappa(\mathbf{A}) = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|$ to get (2).

(b) Set $\delta\mathbf{x} = \tilde{\mathbf{x}} - \mathbf{x}_\star$. Then we know that

$$\big(\mathbf{A} + \delta\mathbf{A}\big)\big(\mathbf{x}_\star + \delta\mathbf{x}\big) = \mathbf{b}.$$

Multiplying things out, we get

$$\mathbf{A}\mathbf{x}_\star + (\delta\mathbf{A})\mathbf{x}_\star + \mathbf{A}(\delta\mathbf{x}) + (\delta\mathbf{A})(\delta\mathbf{x}) = \mathbf{b}.$$

The term $(\delta\mathbf{A})(\delta\mathbf{x})$ is quadratic in ε, so using that $\mathbf{A}\mathbf{x}_\star = \mathbf{b}$, we find that

$$\mathbf{A}(\delta\mathbf{x}) = -(\delta\mathbf{A})\mathbf{x}_\star + O(\varepsilon^2).$$

Then

$$\delta\mathbf{x} = -\mathbf{A}^{-1}(\delta\mathbf{A})\mathbf{x}_\star + O(\varepsilon^2),$$

and the estimate follows immediately by taking norms and dividing by $\|\mathbf{x}_\star\|$.

Note: The solution given above follows the book and the lectures. To be absolutely strict, one should actually *prove* that $(\delta\mathbf{A})(\delta\mathbf{x})$ is a quadratic term in ε. This is *not* necessary to get full marks, but a stickler for mathematical logic may prefer the following proof:

$$\tilde{\mathbf{x}} - \mathbf{x}_\star = \big(\mathbf{A} + \delta\mathbf{A}\big)^{-1}\mathbf{b} - \mathbf{x}_\star = \big(\mathbf{I} + \mathbf{A}^{-1}(\delta\mathbf{A})\big)^{-1}\mathbf{A}^{-1}\mathbf{b} - \mathbf{x}_\star = \big(\mathbf{I} + \mathbf{A}^{-1}(\delta\mathbf{A})\big)^{-1}\mathbf{x}_\star - \mathbf{x}_\star.$$

You can prove that if $\mathbf{T}$ is a small matrix, then

$$\big(\mathbf{I} + \mathbf{T}\big)^{-1} = \mathbf{I} - \mathbf{T} + O(\|\mathbf{T}\|^2).$$

(Use for instance that $(\mathbf{I} + \mathbf{T})^{-1} = \sum_{n=0}^{\infty}(-\mathbf{T})^n$.) You then get

$$\tilde{\mathbf{x}} - \mathbf{x}_\star = -\mathbf{A}^{-1}(\delta\mathbf{A})\mathbf{x}_\star + O(\|\delta\mathbf{A}\|^2)$$

and you take norms, etc.

Question 6: (15p) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{b} \in \mathbb{R}^{m\times 1}$. After two steps of the Arnoldi process, starting with $\mathbf{q}_1 = \mathbf{b}/\|\mathbf{b}\|$, as usual, we find that the following factorization holds *exactly*:

$$\mathbf{A}\big[\mathbf{q}_1\ \mathbf{q}_2\big] = \big[\mathbf{q}_1\ \mathbf{q}_2\big]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

In other words, we ended up with the nontypical (but lucky!) case where $h_{32} = 0$.

(a) (5p) Prove that the linear space $V = \text{span}(\mathbf{q}_1, \mathbf{q}_2)$ is an invariant subspace of $\mathbf{A}$. (Recall that a linear space W is an *invariant subspace* of a matrix $\mathbf{A}$ if $\mathbf{A}\mathbf{x} \in W$ whenever $\mathbf{x} \in W$.)

(b) (5p) Prove that the two distinct eigenvalues of $\mathbf{H}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are also eigenvalues of $\mathbf{A}$.

(c) (5p) Specify the eigenvectors of $\mathbf{A}$ associated with the two eigenvalues of $\mathbf{H}_2$. These will be expressed as linear combinations of $\mathbf{q}_1$ and $\mathbf{q}_2$.

Solution: Observe that the given relation simply means that

$$\mathbf{A}\mathbf{q}_1 = 2\mathbf{q}_1 + \mathbf{q}_2 \qquad \text{and} \qquad \mathbf{A}\mathbf{q}_2 = \mathbf{q}_1 + 2\mathbf{q}_1.$$

(a) Suppose that $\mathbf{x} \in V$. Then $\mathbf{x} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2$. We find that

$$\mathbf{A}\mathbf{x} = c_1\mathbf{A}\mathbf{q}_1 + c_2\mathbf{A}\mathbf{q}_2 = c_1(2\mathbf{q}_1 + \mathbf{q}_2) + c_2(\mathbf{q}_1 + 2\mathbf{q}_2) = (2c_1 + c_2)\mathbf{q}_1 + (c_1 + 2c_2)\mathbf{q}_2 \in V.$$

Alt. solution: Say $\mathbf{x} \in V$. Then $\mathbf{x} = \mathbf{Q}_1\mathbf{y}$ where $\mathbf{Q}_1 = [\mathbf{q}_1, \mathbf{q}_2]$. Then $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{Q}_1\mathbf{y} = \mathbf{Q}_1\mathbf{H}\mathbf{y} \in V$.

(b,c) One easily finds that the matrix $\mathbf{H}_2$ has the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 1$, and that these are associated with the eigenvectors $\mathbf{v}_1 = [1, 1]^*$ and $\mathbf{v}_2 = [1, -1]^*$, respectively. To build an eigenvector for $\mathbf{A}$ with eigenvalue 3, we use the entries of $\mathbf{v}_1$ to form a linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$ to set

$$\mathbf{w}_1 = \mathbf{q}_1 + \mathbf{q}_2.$$

Then

$$\mathbf{A}\mathbf{w}_1 = \mathbf{A}\mathbf{q}_1 + \mathbf{A}\mathbf{q}_2 = 2\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_1 + 2\mathbf{q}_2 = 3\mathbf{q}_1 + 3\mathbf{q}_2 = 3\mathbf{w}_1.$$

So $\mathbf{w}_1$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = 3$. Setting $\mathbf{w}_2 = \mathbf{q}_1 - \mathbf{q}_2$, an analogous computation shows that $\mathbf{A}\mathbf{w}_2 = \mathbf{w}_2$.

Alt. solution: Say $\mathbf{H}_2\mathbf{v} = \lambda\mathbf{v}$. Then $\mathbf{A}\mathbf{Q}_1\mathbf{v} = \mathbf{Q}_1\mathbf{H}_2\mathbf{v} = \lambda\mathbf{Q}_1\mathbf{v}$ so $\mathbf{Q}_1\mathbf{v}$ is an evec of $\mathbf{A}$.

Note: One can approach things as a matrix factorization problem, but this is harder since you then have to extend $\{\mathbf{q}_1, \mathbf{q}_2\}$ to a full basis. To be precise, it would go something like this: Set $\mathbf{Q}_1 = [\mathbf{q}_1, \mathbf{q}_2]$ and then extend $\mathbf{Q}_1$ to a full unitary matrix $\mathbf{Q} = [\mathbf{Q}_1 \quad \mathbf{Q}_2]$. Then $\mathbf{A}$ admits the factorization

$$\mathbf{A} = \big[\mathbf{Q}_1 \quad \mathbf{Q}_2\big]\begin{bmatrix} \mathbf{H}_2 & \times \\ \mathbf{0} & \times \end{bmatrix}\begin{bmatrix} \mathbf{Q}_1^* \\ \mathbf{Q}_2^* \end{bmatrix}.$$

Computing the eigenvalue decomposition of $\mathbf{H}_2$, we get

$$\mathbf{H}_2 = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^*, \qquad \text{where} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \quad \mathbf{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Now

$$\mathbf{A} = \big[\mathbf{Q}_1 \quad \mathbf{Q}_2\big]\begin{bmatrix} \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^* & \times \\ \mathbf{0} & \times \end{bmatrix}\begin{bmatrix} \mathbf{Q}_1^* \\ \mathbf{Q}_2^* \end{bmatrix} = \big[\mathbf{Q}_1\mathbf{V} \quad \mathbf{Q}_2\big]\begin{bmatrix} \boldsymbol{\Lambda} & \times \\ \mathbf{0} & \times \end{bmatrix}\begin{bmatrix} (\mathbf{Q}_1\mathbf{V})^* \\ \mathbf{Q}_2^* \end{bmatrix}.$$

It is now obvious the $\mathbf{A}$ has λ_1 and λ_2 as eigenvalues, and that the corresponding eigenvectors are the two columns of $\mathbf{Q}_1\mathbf{V}$.

Note that the extension of the basis is necessary. Many students attempted a solution where one would introduce an orthonormal matrix $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2]$, so that the given relation can be written $\mathbf{A}\mathbf{Q} = \mathbf{Q}\mathbf{H}_2$. But in this case, $\mathbf{Q}$ is not square, it is not unitary, $\mathbf{Q}^{-1}$ does not exist, so it does *not* follow that $\mathbf{A} = \mathbf{Q}\mathbf{H}_2\mathbf{Q}^*$. (In fact, the relation $\mathbf{A} = \mathbf{Q}\mathbf{H}_2\mathbf{Q}^*$ would imply that $\mathbf{A}$ has rank 2.)